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Global solutions for nonlinear wave equations with localized dissipations in exterior domains

Makoto Nakamura

Mathematical Institute, Tohoku University, Aoba, Sendai, 980-8578, Japan

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ABSTRACT

The Cauchy problem for nonlinear wave equations with localized dissipation is considered in exterior domains outside of compact obstacles in three spatial dimensions. Under the null conditions for the quadratic nonlinear terms, the global solutions are shown for sufficiently small data. The solutions which have different propagation speeds are considered.

1. Introduction

Let $\mathcal{K}$ be any fixed compact domain in three dimensional Euclidean space $\mathbb{R}^3$ with smooth boundary. Without loss of generality, we assume that $0 \in \mathcal{K} \subset \{x \in \mathbb{R}^3 \colon |x| < 1\}$ by the shift and scaling argument. Let $a = a(x)$ be a nonnegative smooth function on $\mathbb{R}^3$ with compact support in $\{x \in \mathbb{R}^3 \colon |x| < 1\}$. We assume that there exists a positive constant C for which $a \geqslant C$ on a domain which contains the closure of $\{x \in \partial\mathcal{K} \colon x \cdot \nu(x) < 0\}$, where $\nu(x)$ is the outward unit normal to $\mathcal{K}$ at a point $x \in \partial\mathcal{K}$. This assumption means that we can consider any shape for the boundary of the obstacle, but we expect the local effect expressed by a in the neighborhood of the boundary where it is not star-shaped.

Let Δ be the Laplacian $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$. We consider the Cauchy problem for a system of nonlinear wave equations with localized dissipation $a(x)\partial_t$ and $D \geqslant 1$ propagation speeds $\{c_I\}_{1 \leqslant I \leqslant D}$, $c_I > 0$,

E-mail address: makoto@math.tohoku.ac.jp.

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$$\begin{cases} \left(\partial_t^2 - c_I^2\Delta + a(x)\partial_t\right)u_I(t,x) = F_I\left(u,u',u''\right)(t,x) & \text{for } t \in [0,\infty),\ x \in \mathbb{R}^3 \backslash \mathcal{K}; \\ u_I(t,x)|_{x\in\partial\mathcal{K}} = 0 & \text{for } t \in [0,\infty); \\ u_I(0,x) = f_I(x), \qquad \partial_t u_I(0,x) = g_I(x) & \text{for } x \in \mathbb{R}^3 \backslash \mathcal{K}, \end{cases} \tag{1.1}$$

where $1 \leqslant I \leqslant D$, we denote $(u_1,\dots,u_D)$ by u, the first derivatives $\{\partial_j u\}_{0\leqslant j\leqslant 3}$ by u', and the second derivatives $\{\partial_j\partial_k u\}_{0\leqslant j,k\leqslant 3}$ by u'' with $\partial_0 = \partial_t$. We denote the nonlinear terms $(F_1,\dots,F_D)$ by F, and we assume that F vanishes to second order and has the form

$$F_I\left(u,u',u''\right) = B_I\left(u'\right) + Q_I\left(u',u''\right) + P_I\left(u,u'\right) + R_I\left(u,u',u''\right), \tag{1.2}$$

where

$$B_I\left(u'\right) := \sum_{\substack{1\leqslant J,K\leqslant D\\ 0\leqslant j,k\leqslant 3}} B_I^{JKjk}\partial_j u_J \partial_k u_K, \tag{1.3}$$

$$Q_I\left(u',u''\right) := \sum_{\substack{1\leqslant J,K\leqslant D\\ 0\leqslant j,k,l\leqslant 3}} Q_I^{JKjkl}\partial_j u_J \partial_k\partial_l u_K, \tag{1.4}$$

$$R_I\left(u,u',u''\right) := \sum_{\substack{1\leqslant K\leqslant D\\ 0\leqslant k,l\leqslant 3}} R_I^{Kkl}\left(u,u'\right)\partial_k\partial_l u_K \tag{1.5}$$

and $R_I^{Kkl}(u,u') = O(|u|^2+|u'|^2)$ and $P_I(u,u') = O(|u|^3+|u'|^3)$ near $(u,u')=0$. Here, $\{B_I^{JKjk}\}_{\substack{1\leqslant I,J,K\leqslant D\\ 0\leqslant j,k\leqslant 3}}$ and $\{Q_I^{JKjkl}\}_{\substack{1\leqslant I,J,K\leqslant D\\ 0\leqslant j,k,l\leqslant 3}}$ are real constants and $\{R_I^{Kkl}(u,u')\}_{\substack{1\leqslant I,K\leqslant D\\ 0\leqslant k,l\leqslant 3}}$ are real-valued polynomials which satisfy the symmetry conditions

$$B_I^{JKjk} = B_I^{JKkj} = B_I^{KJjk}, \tag{1.6}$$

$$Q_I^{JKjkl} = Q_I^{JKjlk} = Q_I^{JKkjl} = Q_I^{KJjkl}, \tag{1.7}$$

$$R_I^{Kkl}\left(u,u'\right) = R_I^{Klk}\left(u,u'\right). \tag{1.8}$$

These symmetry conditions are used to derive the energy estimates.

We assume that the propagation speeds $\{c_I\}_{1\leqslant I\leqslant D}$ are positive and distinct with $0 < c_1 < \cdots < c_D$ for convenience. Straightforward modifications can be made to allow variant components to have the same speed. To show the global solutions of our problem, we also assume that the quadratic terms satisfy the null conditions

$$\sum_{0\leqslant j,k\leqslant 3} B_I^{IIjk}\xi_j\xi_k = \sum_{0\leqslant j,k,l\leqslant 3} Q_I^{IIjkl}\xi_j\xi_k\xi_l = 0 \tag{1.9}$$

for any $1 \leqslant I \leqslant D$ and any $(\xi_0,\xi_1,\xi_2,\xi_3) \in \mathbb{R}^4$ which satisfy $\xi_0^2 = c_I^2(\xi_1^2+\xi_2^2+\xi_3^2)$.

Since (1.1) is the initial and boundary value problem, the initial data $f = (f_1,\dots,f_D)$ and $g = (g_1,\dots,g_D)$ must satisfy the following compatibility conditions. For any natural number k, let $\overline{\partial_x}^k u := \{\partial_x^\alpha u \mid |\alpha| \leqslant k\}$. For the solution u of (1.1), we can write $\partial_t^k u(0,\cdot) = \psi_k(\overline{\partial_x}^k f, \overline{\partial_x}^{k-1} g)$, where ψ_k is called the compatibility function, which depends on f and g and F. Our Dirichlet condition requires some conditions on the compatibility functions. Since we are considering smooth solutions, we require the compatibility conditions of infinite order, namely, $\psi_k(\overline{\partial_x}^k f, \overline{\partial_x}^{k-1} g)|_{\partial\mathcal{K}} = 0$ holds for any $k \geqslant 1$.

Our result in this paper is the following theorem.

Theorem 1.1. *Let $\mathcal{K}$, a and F be as above. Let f and g be smooth functions which satisfy the compatibility conditions of infinite order. Then there exist a positive real number $\varepsilon_0 > 0$ and a positive large natural number N such that if*

$$\sum_{|\alpha|\leqslant N} \big\| \langle x\rangle^{|\alpha|} \partial_x^\alpha f \big\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \sum_{|\alpha|\leqslant N-1} \big\| \langle x\rangle^{|\alpha|+1} \partial_x^\alpha g \big\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \leqslant \varepsilon_0, \tag{1.10}$$

then (1.1) *has a unique global solution $u \in C^\infty([0,\infty) \times \mathbb{R}^3\setminus\mathcal{K})$.*

Throughout this paper, we use the following notations. The Lebesgue space of p order in the exterior domain is denoted by $L^p(\mathbb{R}^3\setminus\mathcal{K})$, or L^p_x or L^p for simplicity. And for any $R > 0$, $L^p(|x| < R)$ denotes $L^p(\{x \in \mathbb{R}^3\setminus\mathcal{K}\colon |x| < R\})$. We put $\|\cdot\|_2 := \|\cdot\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}$. For $c > 0$, $\Box_c := \partial_t^2 - c^2\Delta$ denotes the D'Alembertian with c propagation speed. Since our estimates for $\Box_c$ are easily reduced to the case of $\Box = \Box_1$, we abbreviate the index c when it is not essential in our estimates. The notation $a \lesssim b$ denotes the inequality $a \leqslant Cb$ for a positive constant C which is not essential for our arguments. We put $r = |x|$ for the spatial variable $x \in \mathbb{R}^3$.

We use the method of commuting vector fields introduced by John and Klainerman [8,9,15]. See also Keel, Smith and Sogge [12] for exterior domains. We denote $\partial_0 = \partial_t$, ∂_1, ∂_2, ∂_3 and the angular derivatives $x_j\partial_k - x_k\partial_j$, $1 \leqslant j \neq k \leqslant 3$, by Z, and the scaling operator $t\partial_t + r\partial_r$ by L. These operators have the commuting properties with $\Box_c$ such as

$$\Box_c Z = Z\Box_c, \qquad \Box_c L = (L+2)\Box_c. \tag{1.11}$$

Note that the operator $t\partial_j + x_j\partial_t$ is not commutable with $\Box_c$ except for $c = 1$, and we do not use this. Although these operators are not commutable with $\Box_c + a\partial_t$, we can construct the energy estimates and the pointwise estimates which contain Z and L by the properties that the function $a(x)$ is not negative and has the compact support. This support condition enables us to use the standard cut-off arguments in exterior domains and to show the required estimates. The most important component is the local energy decay estimates by Nakao [23] which is formulated as Lemma 2.1 in this paper. He has considered Kirchhoff-type wave equations [25] and wave equations of power type nonlinearities [24]. See [26] for more references.

When the dissipation is given by $a(x) = 1$ in $\mathbb{R}^3\setminus\mathcal{K}$, Shibata [27] has proved the Cauchy problems has small global solutions regardless of the geometry of $\mathcal{K}$. When $\mathcal{K} = \phi$ and $a(x) = 0$, the problem (1.1) is nonlinear wave equations in $[0,\infty) \times \mathbb{R}^3$ and it is known that any nontrivial solutions blow up in finite time in general for quadratic nonlinearities (see John [7]), while the null conditions guarantee the global solutions (see Christodoulou [1] and Klainerman [16]). In this sense, our problem is critical for global solutions. And we have already shown that the solutions exist almost globally for sufficiently small data in [21] without the null conditions. Here, almost global means that the lifespan of solutions is guaranteed from below by $C\exp(c/\varepsilon)$ with positive constants C, c and an adequate norm of the initial data ε (see Klainerman [14], John and Klainerman [9] for the boundaryless case $[0,\infty) \times \mathbb{R}^3$, and see also Klainerman and Sideris [17] for the multiple speed case on which our estimate Lemma 2.18 is based). In this paper, we show the solutions are global if we put the null conditions on the quadratic nonlinearities.

We note that our conditions for the quadratic terms exclude the term u^2. Du, Ma and Yao have shown that the existence time of solutions for u^2 is longer than C/ε^2 in [2], which is sharp by the result in [6, Theorem III] and the finite propagation speeds of (1.1).

When $a(x) = 0$, the problems have been considered for several types of the obstacles $\mathcal{K}$. There is a series of papers on almost global and global solutions in this case. See Keel, Smith and Sogge [10] for convex obstacles, [12] for nontrapping obstacles, [11] and [13] for star-shaped obstacles. See also [19,20,22] for Ikawa's type trapping obstacles (see Ikawa [4] and [5] for the details on the obstacles).

2. Preparation

In this section, we introduce or show several estimates which are required to prove Theorem 1.1.

2.1. Energy estimates

We use the following energy estimates for quasilinear wave equations. Let f and g be initial data and F be a function. And let ${\gamma_I}^{J\alpha\beta}$, $1 \leqslant I, J \leqslant D$, $0 \leqslant \alpha, \beta \leqslant 3$, be functions which satisfy the symmetry conditions

$$
{\gamma_I}^{J\alpha\beta} = {\gamma_I}^{J\beta\alpha} = {\gamma_J}^{I\alpha\beta}.
$$

We consider the equation

$$
\left(\partial_t^2 - c_I^2 \Delta + a\partial_t\right) u_I = F_I - \sum_{\substack{1 \leqslant J \leqslant D \\ 0 \leqslant \alpha,\beta \leqslant 3}} {\gamma_I}^{J\alpha\beta} \partial_\alpha \partial_\beta u_J \quad \text{for } 1 \leqslant I \leqslant D. \tag{2.1}
$$

We define the energy momentums $e_\alpha(u)$, $0 \leqslant \alpha \leqslant 3$, and the error term $R(u)$ by

$$
\begin{aligned}
e_0(u) &:= \sum_{1 \leqslant I \leqslant D} \left\{ (\partial_t u_I)^2 + c_I^2 |\nabla u_I|^2 \right\} + \sum_{\substack{1 \leqslant I, J \leqslant D \\ 0 \leqslant \beta \leqslant 3}} 2{\gamma_I}^{J0\beta} \partial_0 u_I \partial_\beta u_J \\
&\quad - \sum_{\substack{1 \leqslant I, J \leqslant D \\ 0 \leqslant \alpha,\beta \leqslant 3}} {\gamma_I}^{J\alpha\beta} \partial_\alpha u_I \partial_\beta u_J, \\
e_i(u) &:= - \sum_{1 \leqslant I \leqslant D} 2c_I^2 \partial_0 u_I \partial_i u_I + \sum_{\substack{1 \leqslant I, J \leqslant D \\ 0 \leqslant \beta \leqslant 3}} 2{\gamma_I}^{Ji\beta} \partial_0 u_I \partial_\beta u_J \quad \text{for } 1 \leqslant i \leqslant 3, \\
R(u) &:= \sum_{\substack{1 \leqslant I, J \leqslant D \\ 0 \leqslant \alpha,\beta \leqslant 3}} 2\left(\partial_\alpha {\gamma_I}^{J\alpha\beta}\right) \partial_0 u_I \partial_\beta u_J - \sum_{\substack{1 \leqslant I, J \leqslant D \\ 0 \leqslant \alpha,\beta \leqslant 3}} \left(\partial_0 {\gamma_I}^{J\alpha\beta}\right) \partial_\alpha u_I \partial_\beta u_J.
\end{aligned}
$$

Then the multiplication of $2\partial_t u_I$ to the equation (2.1) shows the divergence form

$$
\partial_t e_0(u) + \operatorname{div}\left(e_1(u), e_2(u), e_3(u)\right) + \sum_{1 \leqslant I \leqslant D} 2a(\partial_t u_I)^2 = \sum_{1 \leqslant I \leqslant D} 2\partial_t u_I F_I + R(u). \tag{2.2}
$$

2.2. Several estimates

Let f and g be initial data and F be an inhomogeneous term. We prepare several estimates for the solution of the problem

$$
\begin{cases}
\left(\partial_t^2 - c_I^2 \Delta + a\partial_t\right) u_I = F_I & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}^3 \setminus \mathcal{K},\ 1 \leqslant I \leqslant D; \\
u|_{\partial\mathcal{K}} = 0 & \text{for } t \in [0,\infty); \\
u(0,\cdot) = f(\cdot), \qquad \partial_t u(0,\cdot) = g(\cdot). &
\end{cases} \tag{2.3}
$$

The most fundamental estimate is the following local energy decay estimates due to Nakao [23] in the case of $M = 0$. The case $M \geqslant 1$ was shown in [21, Lemma 2.2].

Lemma 2.1. *Let $R > 1$, $1 \leqslant I \leqslant D$. If* $\operatorname{supp} F_I \subset [0,\infty) \times \{x \in \mathbb{R}^3 \backslash \mathcal{K}\colon\ |x| < R\}$, $\operatorname{supp} f \cup \operatorname{supp} g \subset \{x \in \mathbb{R}^3 \backslash \mathcal{K}\colon\ |x| < R\}$*. Then there exists a positive constant c such that for any $M \geqslant 0$ and $\mu_0 \geqslant 0$, the solution u of* (2.3) *satisfies*

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\| L^\mu \partial^\alpha u_I'(t,x)\right\|_{L^2(|x|<R)} \lesssim e^{-ct/2} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\| L^\mu \partial^\alpha u_I'(0,x)\right\|_{L^2(|x|<R+1)} + \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int_0^t e^{-c(t-s)/2} \left\| L^\mu \partial^\alpha F_I(s,x)\right\|_{L^2(|x|<R+1)} ds. \tag{2.4}$$

We use the following boundary term estimates to bound the terms from the commutator estimates. See [21, Lemma 2.11] for the proof.

Lemma 2.2. *For any $M \geqslant 0$ and $\mu_0 \geqslant 0$, the solution u of* (2.3) *satisfies*

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int_0^t \left\| L^\mu \partial^\alpha u_I'(s,x)\right\|_{L^2(|x|<4)} ds \\
&\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \left\| \langle x\rangle \big(L^\mu \partial^\alpha u_I\big)(0,\cdot)\right\|_{L^2} + \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int_0^t \left\| L^\mu \partial^\alpha F(\tau,x)\right\|_{L^2(|x|<5)} d\tau \\
&\quad + \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int_0^t \int_0^s \left\| L^\mu \partial^\alpha F_I(\tau,y)\right\|_{L^2(||y|-c_I(s-\tau)|<5)} d\tau\, ds.
\end{aligned} \tag{2.5}$$

One of the key estimates is the following weighted energy estimates by Keel, Smith and Sogge [12]. The local dissipation version has been shown in [21, Proposition 2.9].

Lemma 2.3. *For any $M \geqslant 0$ and $\mu_0 \geqslant 0$, the solution u of* (2.3) *satisfies*

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \big(\log(e+T)\big)^{-1/2} \left\| \langle x\rangle^{-1/2} L^\mu \partial^\alpha u_I'\right\|_{L^2_{t,x}} &\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \left\| L^\mu \partial^\alpha u_I(0,\cdot)\right\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \\
&\quad + \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int_0^T \left\| L^\mu \partial^\alpha F_I(s,\cdot)\right\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} ds \\
&\quad + \sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\mu_0}} \left\| L^\mu \partial^\alpha F_I(t,x)\right\|_{L^2_{t,x}(|x|<4)}.
\end{aligned} \tag{2.6}$$

Here, the above estimate holds with ∂ replaced by Z.

We use the following estimates to bound the nonlinearities which satisfy the null conditions. By these estimates, we can construct the global solutions. See [29,30] for the proof.

Lemma 2.4. *Suppose that the quadratic parts of the nonlinearity $Q(u', u'')$, $B(u')$ satisfy the null conditions* (1.9)*. Then we have the following estimates*

$$\left|\sum_{0\leqslant j,k,l\leqslant 3} Q_I^{IIjkl}\partial_j u\partial_k\partial_l v\right| \leqslant \frac{C}{\langle r\rangle}\cdot\left\{\sum_{\substack{\mu+|\alpha|\leqslant 1\\ \mu\leqslant 1}}\left|L^\mu Z^\alpha u\right|\left|\partial^2 v\right| + \sum_{\substack{\mu+|\alpha|\leqslant 1\\ \mu\leqslant 1}}|\partial u|\left|L^\mu Z^\alpha \partial v\right|\right\} + C\frac{\langle c_I t - r\rangle}{\langle t+r\rangle}|\partial u|\left|\partial^2 v\right|, \tag{2.7}$$

and

$$\left|\sum_{0\leqslant j,k\leqslant 3} B_I^{II,jk}\partial_j u\partial_k v\right| \leqslant \frac{C}{\langle r\rangle}\cdot\left\{\sum_{\substack{\mu+|\alpha|\leqslant 1\\ \mu\leqslant 1}}\left|L^\mu Z^\alpha u\right||\partial v| + |\partial u|\sum_{\substack{\mu+|\alpha|\leqslant 1\\ \mu\leqslant 1}}\left|L^\mu Z^\alpha v\right|\right\} + C\frac{\langle c_I t - r\rangle}{\langle t+r\rangle}|\partial u||\partial v|. \tag{2.8}$$

2.3. L^∞–L^∞ estimates

We use the following pointwise estimates. The first is due to Kubota and Yokoyama [18, Theorem 3.4] for the boundaryless case (see also [20, Theorem 2.4] and its proof).

Proposition 2.5. *For any $\theta > 0$, the solution u of*

$$\begin{cases} \left(\partial_t^2 - c_I^2\Delta\right)u_I = F_I & \text{for } (t,x)\in[0,\infty)\times\mathbb{R}^3,\ 1\leqslant I\leqslant D;\\ u(0,\cdot) = f(\cdot), & \partial_t u(0,\cdot) = g(\cdot) \end{cases} \tag{2.9}$$

satisfies

$$\begin{aligned}(1+t+r)&\left(1+\log\frac{1+t+r}{1+|c_I t - r|}\right)^{-1}\left|u_I(t,x)\right| \\ &\lesssim \sum_{\substack{\mu+|\alpha|\leqslant 3\\ \mu\leqslant 1\\ j\leqslant 1}}\left\|\left(\langle y\rangle\partial\right)^j L^\mu Z^\alpha u_I(0,y)\right\|_{L^2_y} \\ &\quad + \sup_{(s,y)\in D_I(t,r)}|y|\left(1+s+|y|\right)^{1+\theta} z_\theta(s,y)\left|F_I(s,y)\right|,\end{aligned} \tag{2.10}$$

where $r = |x|$ and D_I, z_θ are defined by

$$D_I(t,x) = D_I(t,r) = \left\{(s,y)\in[0,\infty)\times\mathbb{R}^3\colon\ 0\leqslant s\leqslant t,\ \left||x| - c_I(t-s)\right| \leqslant |y| \leqslant |x| + c_I(t-s)\right\},$$

$$\Lambda_I = \left\{(s,y)\in[0,\infty)\times\mathbb{R}^3\colon\ s\geqslant 1,\ |y|\geqslant 1,\ \left||y| - c_I s\right| \leqslant \min_{1\leqslant J,K\leqslant D}|c_J - c_K|s/3\right\},$$

$$z_\theta(s,y) = z_\theta\left(s,|y|\right) = \begin{cases} (1+||y| - c_J s|)^{1-\theta} & \text{if } (s,y)\in\Lambda_J \text{ for } 1\leqslant J\leqslant D;\\ (1+|y|)^{1-\theta} & \text{if } (s,y)\in((0,\infty)\times\mathbb{R}^3)\backslash(\bigcup_{1\leqslant J\leqslant D}\Lambda_J). \end{cases}$$

The second is the exterior domain version of the above proposition.

Proposition 2.6. *For any $M \geqslant 0$, $\mu_0 \geqslant 0$ and any $\theta > 0$, the solution u of* (2.3) *satisfies*

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} (1+t+r)\left(1+\log\frac{1+t+r}{1+|c_I t-r|}\right)^{-1}\left|L^\mu Z^\alpha u_I(t,x)\right| \\
&\quad\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+6\\ \mu\leqslant\mu_0+1\\ j\leqslant 1}} \left\|\big(\langle y\rangle\partial\big)^j L^\mu Z^\alpha u_I(0,y)\right\|_{L^2_y} \\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sup_{(s,y)\in D_I(t,r)} |y|\big(1+s+|y|\big)^{1+\theta} z_\theta(s,y)\left|L^\mu Z^\alpha F_I(s,y)\right| \\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0}} \sup_{0\leqslant s\leqslant t} (1+s)\left\|L^\mu Z^\alpha F_I(s,y)\right\|_{L^2(|y|<3)} \\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t\\ |y|<3}} \sup_{(\tau,\xi)\in D_I(s,y)} |\xi|\big(1+\tau+|\xi|\big)^{1+\theta} z_\theta(\tau,\xi)\left|L^\mu \partial^\alpha F_I(\tau,\xi)\right|, \qquad (2.11)
\end{aligned}$$

where $r = |x|$ and D_I, z_θ are defined by

$$D_I(t,x) = D_I(t,r) = \left\{(s,y)\in[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}\colon\ 0\leqslant s\leqslant t,\ \big||x|-c_I(t-s)\big|\leqslant|y|\leqslant|x|+c_I(t-s)\right\},$$

$$\Lambda_I = \left\{(s,y)\in[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}\colon\ s\geqslant 1,\ |y|\geqslant 1,\ \big||y|-c_I s\big|\leqslant \min_{1\leqslant J,K\leqslant D}|c_J-c_K|s/3\right\},$$

$$z_\theta(s,y) = z_\theta\big(s,|y|\big) = \begin{cases} (1+||y|-c_J s|)^{1-\theta} & \textit{if } (s,y)\in\Lambda_J \textit{ for } 1\leqslant J\leqslant D;\\ (1+|y|)^{1-\theta} & \textit{if } (s,y)\in((0,\infty)\times\mathbb{R}^3\backslash\mathcal{K})\backslash(\bigcup_{1\leqslant J\leqslant D}\Lambda_J). \end{cases}$$

To prove Proposition 2.6, we prepare the following lemmas.

Lemma 2.7. *For any $M \geqslant 0$, $\mu_0 \geqslant 0$ and any $\theta > 0$, the solution u of* (2.3) *satisfies*

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} (1+t+r)\left(1+\log\frac{1+t+r}{1+|c_I t-r|}\right)^{-1}\left|L^\mu Z^\alpha u_I(t,x)\right| \\
&\quad\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0+1\\ j\leqslant 1}} \left\|\big(\langle x\rangle\partial\big)^j L^\mu Z^\alpha u(0,\cdot)\right\|_{L^2} \\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sup_{(s,y)\in D_I(t,r)} |y|\big(1+s+|y|\big)^{1+\theta} z_\theta(s,y)\left|L^\mu Z^\alpha F_I(s,y)\right| \\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t\\ |y|<2}} (1+s)\left|L^\mu \partial^\alpha u_I(s,y)\right|. \qquad (2.12)
\end{aligned}$$

Proof. For $|x| < 2$, the left hand side is bounded by the last term of the right hand side. For $|x| \geqslant 2$, we consider a cut-off function ρ which satisfies $\rho(r) = 1$ for $r \geqslant 2$ and $\rho(r) = 0$ for $r \leqslant 1$. Then we have

$$\Box_c\big(\rho L^\mu Z^\alpha u\big) = \rho \Box_c L^\mu Z^\alpha u - 2c^2(\nabla\rho)\nabla L^\mu Z^\alpha u - c^2(\Delta\rho) L^\mu Z^\alpha u. \tag{2.13}$$

Let u_1 and u_2 be the solutions for the boundaryless case which satisfy

$$\Box_c u_1 = \rho\Box_c L^\mu Z^\alpha u, \qquad u_1(0,\cdot) = \rho L^\mu Z^\alpha u(0,\cdot), \qquad \partial_t u_1(0,\cdot) = \rho\partial_t L^\mu Z^\alpha u(0,\cdot), \tag{2.14}$$

$$\Box_c u_2 = -2c^2(\nabla\rho)\nabla L^\mu Z^\alpha u - c^2(\Delta\rho) L^\mu Z^\alpha u, \qquad u_2(0,\cdot) = 0, \qquad \partial_t u_2(0,\cdot) = 0. \tag{2.15}$$

We note that $\rho L^\mu Z^\alpha u = u_1 + u_2$ holds. For u_1, Proposition 2.5 shows

$$\begin{aligned}
&(1+t+r)\left(1+\log\frac{1+t+r}{1+|c_I t - r|}\right)^{-1}\big|u_1(t,x)\big| \\
&\quad\lesssim \sum_{\substack{\nu+|\beta|\leqslant 3\\ \nu\leqslant 1\\ j\leqslant 1}} \big\|\big(\langle x\rangle\partial\big)^j L^\nu Z^\beta u_1(0,\cdot)\big\|_{L^2} + \sup_{(s,y)\in D(t,r)} |y|\big(1+s+|y|\big)^{1+\theta} z_\theta(s,y)\big|\Box_c u_1(s,y)\big| \\
&\quad\lesssim \sum_{\substack{\nu+|\beta|\leqslant \mu+|\alpha|+3\\ \nu\leqslant\mu+1\\ j\leqslant 1}} \big\|\big(\langle x\rangle\partial\big)^j L^\nu Z^\beta u(0,\cdot)\big\|_{L^2} \\
&\qquad + \sup_{(s,y)\in D(t,r)} |y|\big(1+s+|y|\big)^{1+\theta} z_\theta(s,y) \sum_{\substack{\nu+|\beta|\leqslant\mu+|\alpha|\\ \nu\leqslant\mu}} \big|L^\nu Z^\beta F(s,y)\big|,
\end{aligned} \tag{2.16}$$

where we have used that the supports of ρ and a are disjoint.

Next, we consider the estimates for u_2. By Kirchhoff formula, we have

$$r\big|u_2(t,x)\big| \lesssim \int_0^t \int_{\substack{|c(t-s)-r|<\eta<c(t-s)+r\\ 1<\eta<2}} \big\|\Box_c u_2(s,\eta\theta)\big\|_{L^\infty_\theta}\,\eta\,d\eta. \tag{2.17}$$

By the definition of u_2, this right hand side is bounded by

$$\frac{1}{1+\max\{(ct-r-2)/c, 0\}} \sup_{\substack{0\leqslant s\leqslant t\\ 1<\eta<2}} (1+s) \sum_{\substack{\nu+|\beta|\leqslant\mu+|\alpha|+1\\ \nu\leqslant\mu}} \big\|L^\nu\partial^\beta u(s,\eta\theta)\big\|_{L^\infty_\theta}. \tag{2.18}$$

So that, we obtain

$$(1+t+r)\big|u_2(t,x)\big| \lesssim \sum_{\substack{\nu+|\beta|\leqslant\mu+|\alpha|+1\\ \nu\leqslant\mu}} \sup_{\substack{0\leqslant s\leqslant t\\ 1<\eta<2}} (1+s)\big\|L^\nu\partial^\beta u(s,\eta\theta)\big\|_{L^\infty_\theta}. \tag{2.19}$$

Combining the estimates for u_1 and u_2, we obtain the required estimates. $\Box$

Lemma 2.8. *For any $M \geqslant 0$, $\mu_0 \geqslant 0$ and any $\theta > 0$, the solution u of* (2.3) *satisfies*

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t\\ |y|<2}} (1+s)\left|L^\mu Z^\alpha u_I(s,y)\right| \\
&\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+5\\ \mu\leqslant\mu_0+1\\ j\leqslant 1}} \left\|\big(\langle x\rangle\partial\big)^j L^\mu Z^\alpha u_I(0,\cdot)\right\|_{L^2} + \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \sup_{0\leqslant s\leqslant t} (1+s)\left\|L^\mu \partial^\alpha F_I\right\|_{L^2(|y|<3)} \\
&\quad + \sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t\\ |y|<3\\ (\tau,\xi)\in D(s,y)}} |\xi|\big(1+\tau+|\xi|\big)^{1+\theta} z_\theta(\tau,\xi)\left|L^\mu\partial^\alpha F_I(\tau,\xi)\right|. \qquad (2.20)
\end{aligned}$$

Proof. Let χ be a cut-off function which satisfies $\chi(r)=1$ for $r\leqslant 3$ and $\chi(r)=0$ for $r\geqslant 4$. Let u_1 and u_2 be the functions which satisfy

$$\begin{aligned}
&(\Box_c + a\partial_t)u_1(t,\cdot) = \chi\big(|\cdot|\big)F(t,\cdot), \qquad u_1(t,\cdot)|_{\partial\mathcal{K}} = 0 \quad \text{for } t\geqslant 0,\\
&u_1(0,\cdot) = \chi\big(|\cdot|\big)f(\cdot), \qquad \partial_t u_1(0,\cdot) = \chi\big(|\cdot|\big)g(\cdot), \qquad (2.21)\\
&(\Box_c + a\partial_t)u_2(t,\cdot) = \big(1-\chi\big(|\cdot|\big)\big)F(t,\cdot), \qquad u_2(t,\cdot)|_{\partial\mathcal{K}} = 0 \quad \text{for } t\geqslant 0,\\
&u_2(0,\cdot) = \big(1-\chi\big(|\cdot|\big)\big)f(\cdot), \qquad \partial_t u_2(0,\cdot) = \big(1-\chi\big(|\cdot|\big)\big)g(\cdot). \qquad (2.22)
\end{aligned}$$

Then we note that $u = u_1 + u_2$ holds.

By Sobolev and Poincaré inequalities, we have

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t\\ |y|<2}} (1+s)\left|L^\mu\partial^\alpha u_1(s,y)\right| \lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \sup_{0\leqslant s\leqslant t} (1+s)\left\|L^\mu\partial^\alpha u_1'(s,y)\right\|_{L^2(|y|<5/2)}. \qquad (2.23)$$

So that, by the local energy decay estimates Lemma 2.1 and by the definition of u_1, we obtain

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t\\ |y|<2}} (1+s)\left|L^\mu\partial^\alpha u_1(s,y)\right| &\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \left\|L^\mu\partial^\alpha u'(0,y)\right\|_{L^2(|y|<3)}\\
&\quad + \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \sup_{0\leqslant s\leqslant t} (1+s)\left\|L^\mu\partial^\alpha F(s,y)\right\|_{L^2(|y|<3)}. \qquad (2.24)
\end{aligned}$$

Next, we consider u_2. Let u_3 be the boundaryless solution which satisfies

$$\Box_c u_3(t,\cdot) = (\Box_c + a\partial_t)u_2 \quad \text{for } t\geqslant 0, \qquad u_3(0,\cdot) = u_2(0,\cdot), \qquad \partial_t u_3(0,\cdot) = \partial_t u_2(0,\cdot). \qquad (2.25)$$

And we define $u_4 = u_2 - u_3$ outside of the obstacle. We note u_4 satisfies

$$(\Box_c + a\partial_t)u_4 = -a\partial_t u_3, \qquad u_4(0,\cdot) = \partial_t u_4(0,\cdot) = 0 \qquad (2.26)$$

outside of the obstacle. Let η be a function which satisfies $\eta(r)=1$ for $r\leqslant 2$ and $\eta(r)=0$ for $r\geqslant 3$. We put $\widetilde{u_2}(t,x)=\eta(|x|)u_3(t,x)+u_4(t,x)$. Then $\widetilde{u_2}$ satisfies

$$\begin{cases} (\Box_c + a\partial_t)\widetilde{u_2} = -2c^2(\nabla\eta)\nabla u_3 - c^2(\Delta\eta)u_3; \\ \widetilde{u_2}|_{\partial\mathcal{K}} = 0; \\ \widetilde{u_2}(0,\cdot)=\partial_t\widetilde{u_2}(0,\cdot)=0. \end{cases} \tag{2.27}$$

By $\widetilde{u_2}(t,x)=u_2(t,x)$ for $|x|<2$, and the same argument for (2.23) and (2.24), we have

$$\begin{aligned} &\sum_{\substack{\mu+|\alpha|\leqslant M \\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t \\ |y|<2}} (1+s)\bigl|L^\mu\partial^\alpha u_2(s,y)\bigr| \\ &\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M \\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t \\ |y|<2}} (1+s)\bigl|L^\mu\partial^\alpha \widetilde{u_2}(s,y)\bigr| \\ &\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+1 \\ \mu\leqslant\mu_0}} \bigl\|L^\mu\partial^\alpha u_3'(0,y)\bigr\|_{L^2(|y|<3)} \\ &\quad + \sum_{\substack{\mu+|\alpha|\leqslant M+2 \\ \mu\leqslant\mu_0}} \sup_{0\leqslant s\leqslant t} (1+s)\bigl\|L^\mu\partial^\alpha u_3(s,y)\bigr\|_{L^2(|y|<3)}, \end{aligned} \tag{2.28}$$

where the data term in the right hand side vanishes by the support conditions in (2.22) and the finite propagation speeds. Since u_3 is the boundaryless solution, we can apply the boundaryless version of the L^∞–L^∞ estimates for u_3. Then the last term is bounded by

$$\begin{aligned} &\sum_{\substack{\mu+|\alpha|\leqslant M+5 \\ \mu\leqslant\mu_0+1 \\ j\leqslant 1}} \bigl\|\bigl(\langle x\rangle\partial\bigr)^j L^\mu Z^\alpha u(0,\cdot)\bigr\|_{L^2} \\ &\quad + \sum_{\substack{\mu+|\alpha|\leqslant M+2 \\ \mu\leqslant\mu_0}} \sup_{\substack{0\leqslant s\leqslant t \\ |y|<3 \\ (\tau,\xi)\in D(s,y)}} |\xi|\bigl(1+\tau+|\xi|\bigr)^{1+\theta} z_\theta(\tau,\xi)\bigl|L^\mu\partial^\alpha F(\tau,\xi)\bigr|. \end{aligned} \tag{2.29}$$

Combining the estimates for u_1 and u_2, we obtain the required estimates. □

Proof of Proposition 2.6. The last term in Lemma 2.7 is estimated by Lemma 2.8. Then we obtain the required estimate. □

Next we consider the $1+r$ version of Proposition 2.6 which is given by the following proposition.

Proposition 2.9. *For any $M\geqslant 0$, $\mu_0\geqslant 0$ and any $\theta>0$, the solution u of* (2.3) *satisfies*

$$\sum_{\substack{\mu+|\alpha|\leqslant M \\ \mu\leqslant\mu_0}} (1+r)\left(1+\log\frac{1+t+r}{1+|c_I t-r|}\right)^{-1}\bigl|L^\mu Z^\alpha u_I(t,x)\bigr|$$

$$
\begin{aligned}
&\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+5\\ \mu\leqslant\mu_0\\ j\leqslant 1}} \big\|\big(\langle x\rangle\partial\big)^j L^\mu Z^\alpha u_I(0,\cdot)\big\|_{L^2}\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sup_{(s,y)\in D_I(t,r)} |y|^2\big(1+s+|y|\big)^\theta z_\theta(s,y)\big|L^\mu Z^\alpha F_I(s,y)\big|\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0}} \sup_{\substack{(s',y')\in D_I(s,y)\\ 0<s<t\\ |y|<3}} \big|y'\big|^2\big(1+s'+\big|y'\big|\big)^\theta z_\theta\big(s',y'\big)\big|L^\mu\partial^\alpha F_I\big(s',y'\big)\big|\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0}} \sup_{0<s<t}\big\|L^\mu\partial^\alpha F_I(s,y)\big\|_{L^2(|y|<5)}. \qquad (2.30)
\end{aligned}
$$

The proof of Proposition 2.9 follows from the boundaryless version of it and the analogous arguments for the proof of Proposition 2.6. We prove the boundaryless version in the following lemma, but we omit the proof of Proposition 2.9.

Lemma 2.10. *For any $\theta > 0$, the solution $\{u\}_{1\leqslant I\leqslant D}$ of*

$$
\begin{cases}
\Box_{c_I} u_I = F_I \quad \text{for } (t,x)\in[0,\infty)\times\mathbb{R}^3;\\
u_I(0,x)=f_I(x), \qquad \partial_t u_I(0,x)=g_I(x) \quad \text{for } x\in\mathbb{R}^3
\end{cases}
\qquad (2.31)
$$

satisfies

$$
\begin{aligned}
&r\left(1+\log\frac{1+t+r}{1+|c_I t-r|}\right)^{-1}\big|u_I(t,x)\big|\\
&\quad\lesssim \sum_{\substack{|\alpha|\leqslant 2\\ j\leqslant 1}} \big\|\big(\langle x\rangle\partial\big)^j Z^\alpha\partial^\alpha u(0,\cdot)\big\|_{L^2}\\
&\qquad+\sup_{(s,y)\in D_I(t,r)} |y|\big(1+|y|\big)\big(1+s+|y|\big)^\theta z_\theta(s,y)\big|F_I(s,y)\big|, \qquad (2.32)
\end{aligned}
$$

where the definition of D_I and z_θ are the same in Proposition 2.5.

We put $c_{D+1} := \min_{1\leqslant k\leqslant D}\{c_k - c_{k-1}\}/3$ and $c_0 := 0$.

Proof. The estimates for the data f_I and g_I have been shown in [20, Lemma 2.2]. We start from the form

$$
u_I(t,x)=\frac{1}{4\pi c_I r}\int_0^t ds\int_{|r-c_I(t-s)|}^{r+c_I(t-s)}\lambda\,d\lambda\int_0^{2\pi}F_I(s,\lambda\Theta_I)\,d\phi, \qquad (2.33)
$$

where $r=|x|$, $A(x)$ is a 3×3 orthogonal matrix which satisfies $x=(0,0,r)A(x)$, and

$$\Theta_I := (\sin\psi_I, \cos\phi, \sin\psi_I \sin\phi, \cos\psi_I) A(x), \tag{2.34}$$

$$\cos\psi_I = \frac{\lambda^2 + r^2 - c_I^2(t-s)^2}{2r\lambda}, \quad 0 \leqslant \psi_I \leqslant \pi, \tag{2.35}$$

which has been shown in [18, p. 140, (3.23)].

By this form, we have

$$\left|u_I(t,x)\right| \leqslant S \cdot \sup_{(s,y)\in D_I(t,r)} |y|\bigl(1+|y|\bigr)\bigl(1+s+|y|\bigr)^{\mu} z_\mu\bigl(s,|y|\bigr)\left|F_I(s,y)\right|, \tag{2.36}$$

where

$$S := \frac{1}{2c_I r}\int_0^t ds \int_{|r-c_I(t-s)|}^{r+c_I(t-s)} (1+\lambda)^{-1}(1+s+\lambda)^{-\mu} z_\mu^{-1}(s,\lambda)\, d\lambda. \tag{2.37}$$

Let $\Lambda_0 := \mathbb{R}^3 \setminus \bigcup_{1\leqslant I\leqslant D} \Lambda_J$. And we put for $0 \leqslant J \leqslant D$

$$S_J := \frac{1}{2c_I r}\int_0^t ds \int_{\substack{|r-c_I(t-s)|<\lambda<r+c_I(t-s)\\ (s,\lambda)\in\Lambda_J}} (1+\lambda)^{-1}(1+s+\lambda)^{-\mu} z_\mu^{-1}(s,\lambda)\, d\lambda. \tag{2.38}$$

When $J \neq 0$, we note that the conditions of the integral region on (s,λ) require

$$a|r-c_I t| \leqslant s \leqslant b(r+c_I t), \tag{2.39}$$

where $a := 1/(c_i + c_j + c_{D+1})$ and $b := 1/(c_i + c_j - c_{D+1})$. So that, we obtain

$$S_J \lesssim \frac{1}{r}\int_{a|r-c_I t|}^{b|r+c_I t|} \frac{1}{1+s}\, ds \lesssim \frac{1}{r}\left(1+\log\frac{1+t+r}{1+|r-c_I t|}\right). \tag{2.40}$$

Next, we consider S_0. We decompose the integral region of λ into $\lambda \geqslant c_1 s/3$ and $\lambda < c_1 s/3$, and we denote them by S_1' and S_2'. For S_1', since we have

$$\begin{aligned} S_0 &\lesssim \frac{1}{r}\int_0^t ds \int_{\substack{|r-c_I(t-s)|\leqslant\lambda\leqslant r+c_I(t-s)\\ \lambda\geqslant c_I s/3}} (1+\lambda)^{-1}(1+s+\lambda)^{-\mu}(1+\lambda)^{-1+\mu}\, d\lambda \\ &\lesssim \frac{1}{r}\int_{|r-c_I t|}^{r+c_I t} (1+\eta)^{-1-\nu}\, d\eta, \end{aligned} \tag{2.41}$$

where we have used the change of variables $\eta = \lambda + c_I s$ to derive the last inequality, we obtain

$$S_1' \lesssim \frac{1}{r}\left(1+\log\frac{1+t+r}{1+|r-c_I t|}\right). \tag{2.42}$$

For S_2', we note that S_2' appears only when $r < c_I t$. And we decompose the integral region of s of S_2' into $(0, (c_I t - r)/c_I)$, $((c_I t - r)/c_I, (1+\delta)(c_I t - r)/c_I)$ and $((1+\delta)(c_I t - r)/c_I, t)$, where δ is a sufficiently small positive number, and we denote them by S_1'', S_2'' and S_3''. We can bound

$$S_1'' \lesssim \frac{1}{r} \int_0^{(c_I t - r)/c_I} \int_{c_I t - r - c_I s}^{c_1 s/3} (1+\lambda)^{-2+\mu}\, d\lambda\, ds \big(1 + |c_I t - r|\big)^{-\mu}, \tag{2.43}$$

$$S_2'' \lesssim \frac{1}{r} \int_{(c_I t - r)/c_I}^{(1+\delta)(c_I t - r)/c_I} \int_{r - c_I t + c_I s}^{c_1 s/3} (1+\lambda)^{-2+\mu}\, d\lambda\, ds \big(1 + |c_I t - r|\big)^{-\mu}. \tag{2.44}$$

And direct calculations show $S_1'' + S_2'' \lesssim \frac{1}{r}$. On the other hand, by the conditions on (s, λ) in the integral region of S_3'', we have

$$S_3'' \lesssim \frac{1}{r} \int_{\delta |c_I t - r|}^{r + c_I t} (1+\lambda)^{-2} \int_{c_I t - r - \lambda \leqslant c_I s \leqslant c_I t - r + \lambda} ds\, d\lambda. \tag{2.45}$$

So that, we obtain

$$S_3'' \lesssim \frac{1}{r} \log \frac{1 + t + r}{1 + |c_I t - r|}. \tag{2.46}$$

Combining these estimates, we obtain (2.40) when $J = 0$. $\square$

2.4. L^∞–L^1 estimates under the Huygens principle

This subsection is devoted to the proof of the following proposition which is the L^∞–L^1 estimates for the solutions. To derive it we use the Huygens principle for the solutions of the boundaryless wave equations (2.48).

Proposition 2.11. *Let $f_I = g_I = 0$ and $\operatorname{supp} F_I \subset \{(s, y)\colon c_1 s/10 \leqslant |y| \leqslant 10 c_D,\ s \geqslant 1\}$ in (2.3). Then for any $M \geqslant 0$ and $\mu_0 \geqslant 0$, the solution u satisfies*

$$\begin{aligned}
&\sum_{\substack{\mu + |\alpha| \leqslant M \\ \mu \leqslant \mu_0}} \big(1 + t + |x|\big) \big|L^\mu Z^\alpha u_I(t, x)\big| \\
&\lesssim \sum_{\substack{\mu + |\alpha| \leqslant M+3 \\ \mu \leqslant \mu_0 + 1}} \sup_{0 \leqslant s \leqslant t} \int_{\mathbb{R}^3 \setminus \mathcal{K}} \big|L^\mu Z^\alpha F_I(s, y)\big|\, dy \left(1 + \left|\log \frac{1+t}{1 + |c_I t - |x||}\right|\right) \\
&\quad + \sum_{\substack{\mu + |\alpha| \leqslant M+7 \\ \mu \leqslant \mu_0 + 1}} \sup_{0 \leqslant s \leqslant t} \int_{\mathbb{R}^3 \setminus \mathcal{K}} \big|L^\mu Z^\alpha F_I(s, y)\big|\, dy \\
&\quad + \sum_{\substack{\mu + |\alpha| \leqslant M+3 \\ \mu \leqslant \mu_0}} \sup_{0 \leqslant s \leqslant t} (1 + s) \big\|L^\mu \partial^\alpha F_I(s, y)\big\|_{L^2(|y| < 4)}.
\end{aligned} \tag{2.47}$$

To prove Proposition 2.11, we prepare several lemmas. In the following, we drop I of u_I, F_I, f_I, g_I for convenience.

Lemma 2.12. *Let u be the solution of the boundaryless equations*

$$\begin{cases} \left(\partial_t^2 - c_I^2\Delta\right)u = F & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}^3; \\ u(t,x) = 0 & \text{for } t \leqslant 0,\ x \in \mathbb{R}^3, \end{cases} \tag{2.48}$$

where

$$\operatorname{supp} F \subset \left\{(s,y):\ c_1 s/10 \leqslant |y| \leqslant 10 c_D s,\ s \geqslant 1\right\}, \quad 0 < c_1 \leqslant \cdots \leqslant c_D.$$

Then

$$\left(1+t+|x|\right)\left|u(t,x)\right| \lesssim \sum_{\substack{\mu+|\alpha|\leqslant 3\\ \mu\leqslant 1}} \sup_{0\leqslant s\leqslant t} \int_{\mathbb{R}^3} \left|L^\mu Z^\alpha F(s,y)\right| dy \left(1 + \left|\log \frac{1+t}{1+|c_I t - |x||}\right|\right). \tag{2.49}$$

Proof. For any fixed (t,x), the value $u(t,x)$ depends only on the values of $F(s,\cdot)$ with

$$s_0 := 2^{-1} \min\left\{1, (10c_D - c_I)^{-1}\right\}\left(1 + \left|c_I t - |x|\right|\right) \leqslant s$$

by the support condition of F and the Huygens principle. We consider a smooth function χ on $\mathbb{R}$ which satisfies $\chi(s) = 0$ for $s \leqslant 1/2$ and $\chi(s) = 1$ for $s \geqslant 1$. Then the solution v of

$$\left(\partial_t^2 - c_I^2\Delta\right)v(s,y) = \chi(s/s_0)F(s,y) \tag{2.50}$$

satisfies $v(t,x) = u(t,x)$. By the L^∞–L^1 estimates (see [13, Proposition 2.1, Lemma 2.2]), we have

$$\left(1+t+|x|\right)\left|v(t,x)\right| \lesssim \sum_{\substack{\mu+|\alpha|\leqslant 3\\ \mu\leqslant 1}} \int_0^t \int_{\mathbb{R}^3} \left|L^\mu Z^\alpha\left(\chi(s/s_0)F(s,y)\right)\right| \frac{dy}{\langle y\rangle}\, ds. \tag{2.51}$$

Since $\langle y\rangle$ and $1+s$ are equivalent and the derivatives of χ are bounded in the support of F, the right hand side is bounded by

$$\int_{s_0/2}^t \frac{ds}{1+s} \sup_{0\leqslant s\leqslant t} \int_{\mathbb{R}^3} \sum_{\substack{\mu+|\alpha|\leqslant 3\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha F(s,y)\right| dy$$

which implies the required inequality. □

We consider the exterior domain version of the above lemma.

Lemma 2.13. *Let u be the solution of*

$$\begin{cases} \left(\partial_t^2 - c_I^2\Delta + a\partial_t\right)u = F & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}^3 \backslash \mathcal{K}; \\ u|_{\partial\mathcal{K}} = 0 & \text{for } t \in [0,\infty); \\ u(t,x) = 0 & \text{for } t \leqslant 0,\ x \in \mathbb{R}^3 \backslash \mathcal{K}, \end{cases} \tag{2.52}$$

where

$$\operatorname{supp} F \subset \bigl\{(s,y)\colon\ c_1 s/10 \leqslant |y| \leqslant 10 c_D s,\ s \geqslant 1\bigr\}, \quad 0 < c_1 \leqslant \cdots \leqslant c_D.$$

Then

$$\begin{aligned}
&\bigl(1+t+|x|\bigr) \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \bigl|L^\mu Z^\alpha u(t,x)\bigr| \\
&\quad\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0+1}} \sup_{0\leqslant s\leqslant t} \int_{\mathbb{R}^3\setminus\mathcal{K}} \bigl|L^\mu Z^\alpha F(s,y)\bigr|\,dy \left(1+\left|\log\frac{1+t}{1+|c_I t-|x||}\right|\right) \\
&\qquad + \sup_{0\leqslant s\leqslant t} \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} (1+s)\bigl\|L^\mu \partial^\alpha u(s,y)\bigr\|_{L^\infty(|y|<2)}.
\end{aligned} \tag{2.53}$$

Proof. When $|x| \leqslant 2$, the left hand side is bounded by the last term of the right hand side. We consider the case $|x| > 2$ in the following. Let η be a function which satisfies $\eta(r) = 0$ for $r \leqslant 1$ and $\eta(r) = 1$ for $r \geqslant 2$. And we consider the solutions u_1 and u_2 of the boundaryless wave equations with zero initial data

$$\begin{aligned}
\Box_{c_I} u_1(s,y) &= \eta\bigl(|y|\bigr)\Box_{c_I} L^\mu Z^\alpha u, \\
\Box_{c_I} u_2(s,y) &= -2c_I^2 \nabla\eta\nabla L^\mu Z^\alpha u - c_I^2 \Delta\eta L^\mu Z^\alpha u.
\end{aligned}$$

Then we have $L^\mu Z^\alpha u = \eta L^\mu Z^\alpha u = u_1 + u_2$ for $|x| > 2$.

The function u_1 is bounded by Lemma 2.12

$$\bigl(1+t+|x|\bigr)\bigl|u_1(t,x)\bigr| \lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0+1}} \sup_{0\leqslant s\leqslant t} \int_{\mathbb{R}^3} \bigl|L^\mu Z^\alpha F(s,y)\bigr|\,dy \left(1+\left|\log\frac{1+t}{1+|c_I t-|x||}\right|\right). \tag{2.54}$$

The function u_2 is bounded by the Kirchhoff formula

$$\begin{aligned}
\bigl|u_2(t,x)\bigr| &\lesssim \frac{1}{r}\int_0^t \int_{|r-c_I(t-s)|}^{r+c_I(t-s)} \bigl\|\Box_{c_I} u_2(s,\rho\theta)\bigr\|_{L^\infty_\theta} \rho\,d\rho\,ds \\
&\lesssim \frac{1}{r}\frac{1}{1+\max\{0,(c_I t-r-2)/c_I\}} \sup_{\substack{0\leqslant s\leqslant t\\ 1\leqslant\rho\leqslant 2}} (1+s)\bigl\|\Box_{c_I} u_2(s,\rho\theta)\bigr\|_{L^\infty_\theta}
\end{aligned} \tag{2.55}$$

since $\operatorname{supp}\Box_{c_I} u_2 \subset \{y \in \mathbb{R}^3\colon\ 1 \leqslant |y| \leqslant 2\}$. So that, we have

$$\bigl(1+t+|x|\bigr)\bigl|u_2(t,x)\bigr| \lesssim \sup_{0\leqslant s\leqslant t} \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} (1+s)\bigl\|L^\mu\partial^\alpha u\bigr\|_{L^\infty(1<|y|<2)}. \tag{2.56}$$

The above estimates for u_1 and u_2 show the required estimates. $\square$

Next we consider the estimate to bound the last term in Lemma 2.13.

Lemma 2.14. *Let u be the solution of*

$$\begin{cases} \left(\partial_t^2 - c_I^2\Delta + a\partial_t\right)u = F & \textit{for } (t,x) \in [0,\infty) \times \mathbb{R}^3 \backslash \mathcal{K}; \\ u|_{\partial\mathcal{K}} = 0 & \textit{for } t \in [0,\infty); \\ u(t,\cdot) = 0 & \textit{for } t \leqslant 0, \end{cases} \tag{2.57}$$

where

$$\operatorname{supp} F \subset \left\{(s,y)\colon\ s \geqslant 0,\ |y| \leqslant 4\right\}.$$

Then

$$\begin{aligned} &(1+t) \sum_{\substack{\mu+|\alpha|\leqslant M \\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha u(t,x)\right\|_{L^2(|x|<4)} \\ &\quad\lesssim \sup_{0\leqslant s\leqslant t} (1+s) \sum_{\substack{\mu+|\alpha|\leqslant \max\{M,\mu_0+1\} \\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha F(s,\cdot)\right\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}. \end{aligned} \tag{2.58}$$

Proof. By the Dirichlet condition and the Poincaré inequality, we have

$$\sum_{\substack{\mu+|\alpha|\leqslant M \\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha u\right\|_{L^2(|x|<4)} \lesssim \sum_{\substack{\mu+|\alpha|\leqslant \max\{M-1,\mu_0\} \\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha u'\right\|_{L^2(|x|<4)}. \tag{2.59}$$

Then by the local energy decay estimates Lemma 2.1, the right hand side is bounded by

$$\sum_{\substack{\mu+|\alpha|\leqslant \max\{M-1,\mu_0\}-1 \\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha F\right\|_{L^2} + \sum_{\substack{\mu+|\alpha|\leqslant \max\{M-1,\mu_0\}+1 \\ \mu\leqslant\mu_0}} \int_0^t e^{-c(t-s)} \left\|L^\mu \partial^\alpha F(s,\cdot)\right\|_{L^2} ds. \tag{2.60}$$

Combining these two estimates, we obtain the required result. $\square$

Lemma 2.15. *Let u be the solution of*

$$\begin{cases} \left(\partial_t^2 - c_I^2\Delta + a\partial_t\right)u = F & \textit{for } (t,x) \in [0,\infty) \times \mathbb{R}^3 \backslash \mathcal{K}; \\ u|_{\partial\mathcal{K}} = 0 & \textit{for } t \in [0,\infty); \\ u(t,x) = 0 & \textit{for } t \leqslant 0,\ x \in \mathbb{R}^3\backslash\mathcal{K}, \end{cases} \tag{2.61}$$

where

$$\operatorname{supp} F \subset \left\{(s,y)\colon\ c_1 s/10 \leqslant |y| \leqslant 10 c_D s,\ s \geqslant 1\right\}, \quad 0 < c_1 \leqslant \cdots \leqslant c_D.$$

Then

$$(1+t)\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u(t,x)\right\|_{L^2(|x|<2)}$$
$$\lesssim \sup_{0\leqslant s\leqslant t}\sum_{\substack{\mu+|\alpha|\leqslant\max\{M,\mu_0+1\}+4\\ \mu\leqslant\mu_0+1}}\int_{\mathbb{R}^3\setminus\mathcal{K}}\left|L^\mu Z^\alpha F(s,y)\right|dy$$
$$+\sup_{0\leqslant s\leqslant t}(1+s)\sum_{\substack{\mu+|\alpha|\leqslant\max\{M,\mu_0+1\}\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha F(s,y)\right\|_{L^2(|y|<4)}. \tag{2.62}$$

Proof. Let χ be a function which satisfies $\chi(r)=1$ for $r\leqslant 3$ and $\chi(r)=0$ for $r\geqslant 4$. We consider the solutions u_1 and u_2 of the wave equations with Dirichlet condition and zero data

$$(\Box_{c_I}+a\partial_t)u_1(t,x)=\chi\big(|x|\big)F(t,x), \tag{2.63}$$
$$(\Box_{c_I}+a\partial_t)u_2(t,x)=\big(1-\chi\big(|x|\big)\big)F(t,x). \tag{2.64}$$

Then we have $u=u_1+u_2$.

The function u_1 is bounded by Lemma 2.14

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}(1+t)\left\|L^\mu\partial^\alpha u_1(t,\cdot)\right\|_{L^2(|x|<2)}\lesssim\sup_{0\leqslant s\leqslant t}(1+s)\sum_{\substack{\mu+|\alpha|\leqslant\max\{M,\mu_0+1\}\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha F(s,y)\right\|_{L^2}. \tag{2.65}$$

For the estimate of u_2, we consider the boundaryless solution u_3 of $\Box_{c_I}u_3=(\Box_{c_I}+a\partial_t)u_2$ with zero initial data, and we put $u_4=u_2-u_3$. Let η be a function which satisfies $\eta(r)=1$ for $r\leqslant 2$ and $\eta(r)=0$ for $r\geqslant 3$. We put $\tilde{u}_2(t,x)=\eta(|x|)u_3(t,x)+u_4(t,x)$. Then $\tilde{u}_2$ satisfies

$$\tilde{u}_2=u_2\quad\text{for } |x|<2, \tag{2.66}$$
$$\tilde{u}_2|_{\partial\mathcal{K}}=0, \tag{2.67}$$
$$(\Box_{c_I}+a\partial_t)\tilde{u}_2=-2c_I\nabla\eta\nabla u_3-c_I^2\Delta\eta u_3, \tag{2.68}$$
$$\operatorname{supp}(\Box_{c_I}+a\partial_t)\tilde{u}_2\subset\big\{x\in\mathbb{R}^3\setminus\mathcal{K}\ \big|\ 2\leqslant|x|\leqslant 3\big\}. \tag{2.69}$$

So that, by Lemma 2.14, we have

$$(1+t)\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u_2\right\|_{L^2(|x|<2)}$$
$$=(1+t)\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha\tilde{u}_2\right\|_{L^2(|x|<2)}$$
$$\lesssim\sup_{0\leqslant s\leqslant t}(1+s)\sum_{\substack{\mu+|\alpha|\leqslant\max\{M,\mu_0+1\}\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha(\Box_{c_I}+a\partial_t)\tilde{u}_2\right\|_{L^2}$$
$$\lesssim\sup_{\substack{0\leqslant s\leqslant t\\ 2\leqslant|y|\leqslant 3}}\sum_{\substack{\mu+|\alpha|\leqslant\max\{M,\mu_0+1\}+1\\ \mu\leqslant\mu_0}}\big(1+s+|y|\big)\left|L^\mu\partial^\alpha u_3(s,y)\right|. \tag{2.70}$$

By Lemma 2.12 and the definition of u_3, the last term is bounded by

$$\sup_{0\leqslant s\leqslant t}\sum_{\substack{\mu+|\alpha|\leqslant\max\{M,\mu_0+1\}+4\\ \mu\leqslant\mu_0+1}}\int_{\mathbb{R}^3\setminus\mathcal{K}}\left|L^\mu Z^\alpha F(s,y)\right|dy. \tag{2.71}$$

Combining the above estimates, we obtain the required results. □

Proof of Proposition 2.11. The last term in Lemma 2.13 is bounded by

$$\sup_{0\leqslant s\leqslant t}\sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0}}(1+s)\left\|L^\mu\partial^\alpha u(s,y)\right\|_{L^2(|y|<2)} \tag{2.72}$$

by the Sobolev embedding $H^2(\mathbb{R}^3\setminus\mathcal{K})\hookrightarrow L^\infty(\mathbb{R}^3\setminus\mathcal{K})$. Then Lemma 2.15 shows the required estimate. □

We use the following $(1+|x|)$ version of Proposition 2.11.

Proposition 2.16. *For any $M\geqslant 0$ and $\mu_0\geqslant 0$, the solution u of* (2.3) *satisfies*

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big(1+|x|\big)\left|L^\mu Z^\alpha u_I(t,x)\right|\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M+5\\ \mu\leqslant\mu_0\\ j\leqslant 1}}\left\|\big(\langle y\rangle\partial\big)^j L^\mu Z^\alpha u_I(0,y)\right\|_{L^2_y}$$
$$+\sum_{\substack{\mu+|\alpha|\leqslant M+5\\ \mu\leqslant\mu_0}}\int_0^t\int_{\mathbb{R}^3\setminus\mathcal{K}}\left|L^\mu Z^\alpha F_I(s,y)\right|\frac{dy}{|y|}\,ds. \tag{2.73}$$

Proof. We use the boundaryless version of the required estimate

$$\big(1+|x|\big)\left|u(t,x)\right|\lesssim\sum_{\substack{|\alpha|\leqslant 2\\ j\leqslant 1}}\left\|\big(\langle x\rangle\partial\big)^j Z^\alpha u(0,\cdot)\right\|_{L^2}$$
$$+\sum_{|\alpha|\leqslant 2}\int_0^t\int_{\mathbb{R}^3\setminus\mathcal{K}}\left|Z^\alpha\Box_c u(s,y)\right|\frac{dy}{1+|y|}\,ds, \tag{2.74}$$

which follows from the analogous argument in [13, Lemma 2.2]. We obtain the required estimate based on the analogous arguments for the proof of Proposition 2.11. The estimate when $a=0$ has been shown in [20, Theorem 4.8]. We omit the details. □

2.5. Sobolev type estimates

We use the following weighted Sobolev estimate from [16]. To prove the estimate, one simply applies Sobolev estimates on $(0,\infty)\times S^2$. The decay results are from comparing the volume elements of $(0,\infty)\times S^2$ and $\mathbb{R}^3$.

Lemma 2.17. *Suppose that $h \in C^\infty(\mathbb{R}^3 \backslash \mathcal{K})$. Then, for $R \geqslant 1$,*

$$\|h\|_{L^\infty(R/2<|x|<R)} \lesssim R^{-1} \sum_{|\alpha|\leqslant 2} \|Z^\alpha h\|_{L^2(R/4<|x|<2R)}. \tag{2.75}$$

And we also use the following estimate by which we can estimate the decay of the solutions away from the light cone. This estimate is an exterior domain analogue of the results of Klainerman and Sideris [17], Sideris [28, Lemma 3.3], Hidano and Yokoyama [3, Lemma 2.1].

Lemma 2.18. *For any $M \geqslant 0$ and $\mu_0 \geqslant 0$, the solution u of (2.3) satisfies*

$$\begin{aligned} &\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \big(|x|^{1/2}\langle c_I t-|x|\rangle + |x|\langle c_I t-|x|\rangle^{1/2}\big)\big|L^\mu Z^\alpha u'(t,x)\big| \\ &\quad\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0+1}} \big\|L^\mu Z^\alpha u'(t,\cdot)\big\|_2 + \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \big\|\langle t+|y|\rangle L^\mu Z^\alpha F(t,y)\big\|_{L^2_y} \\ &\qquad + \sum_{\mu\leqslant\mu_0} (1+t)\big\|L^\mu u'(t,y)\big\|_{L^2(|y|<2)}. \end{aligned} \tag{2.76}$$

Proof. The required estimate has been shown in [19, Lemmas 4.2 and 4.3] with F is replaced by $\Box_{c_I} u$. So that, it suffices to show

$$\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \big\|\langle t+|y|\rangle L^\mu Z^\alpha (a\partial_t u)(t,y)\big\|_{L^2_y} \tag{2.77}$$

is bounded by the right hand side of the required estimate. Since the support of a is in $\{x \in \mathbb{R}^3 \backslash \mathcal{K}\colon |x| < 1\}$, the above term is bounded by

$$\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0+1}} \big\|L^\mu Z^\alpha u'(t,\cdot)\big\|_2 + \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \big\|t\partial_t L^\mu \partial^\alpha u(t,y)\big\|_{L^2(|y|<1)}. \tag{2.78}$$

Since this second term is bounded by $\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0+1}} \|L^\mu \partial^\alpha u(t,y)\|_{L^2(|y|<1)}$ by $L = t\partial + r\partial_r$, it is bounded by

$$\sum_{\mu\leqslant\mu_0+1} \big\|L^\mu u(t,y)\big\|_{L^2(|y|<1)} + \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0+1}} \big\|L^\mu \partial^\alpha u'(t,y)\big\|_{L^2(|y|<1)}. \tag{2.79}$$

By the Poincaré inequality, we have

$$\sum_{\mu\leqslant\mu_0+1} \big\|L^\mu u(t,y)\big\|_{L^2(|y|<1)} \lesssim \big\|u'(t,\cdot)\big\|_{L^2(|x|<2)} + \sum_{\mu\leqslant\mu_0} (1+t)\big\|L^\mu u'(t,y)\big\|_{L^2(|y|<1)}. \tag{2.80}$$

Therefore we obtain the required estimate. $\square$

2.6. Commutator estimates

Let χ be a nonnegative function with $\chi(r) = 0$ if $r \leqslant 1$ and $\chi(r) = 1$ if $r \geqslant 2$. We put

$$\tilde{L} := t\partial_t + \chi(r)r\partial_r. \tag{2.81}$$

When we consider the higher derivatives by L and Z of the solution u_I of (2.1), we use the following commutators:

$$\Box_{c_I} L = (L+2)\Box_{c_I}, \qquad \Box_{c_I} Z = Z\Box_{c_I}. \tag{2.82}$$

More detailed relations are

$$\begin{aligned}
&\left(\partial_t^2 - c_I\Delta + a\partial_t\right)\tilde{L}^l\partial_t^m u_I + \sum_{\substack{1\leqslant J\leqslant D\\ 0\leqslant\alpha,\beta\leqslant 3}} {\gamma_I}_J{}^{\alpha\beta}\partial_\alpha\partial_\beta\tilde{L}^l\partial_t^m u_J\\
&= (\tilde{L}+2)^l\partial_t^m F_I + \sum_{\substack{p\leqslant l-1\\ |\nu|\leqslant 1}} \chi_{k,\nu}\tilde{L}^p\partial_t^m\partial^\nu\partial_x u_I + \sum_{\substack{p+q\leqslant l\\ r+s=m\\ q+s\leqslant l+m-1}} C_{p,q,r,s}\cdot\left(\tilde{L}^p\partial_t^r a\right)\tilde{L}^q\partial_t^s\partial_t u_I\\
&\quad + \sum_{J=1}^{D}\sum_{\substack{p\leqslant l-1\\ |\nu|\leqslant 1\\ 0\leqslant\alpha,\beta\leqslant 3}} \chi_{k,\nu,J}{\gamma_I}^{J\alpha\beta}\tilde{L}^p\partial_t^m\partial^\nu\partial_x u_J\\
&\quad + \sum_{J=1}^{D}\sum_{\substack{p+q\leqslant l\\ r+s=m\\ p+r\geqslant 1\\ 0\leqslant\alpha,\beta\leqslant 3}} C_{p,q,r,s,J}\cdot\left(\tilde{L}^p\partial_t^r{\gamma_I}^{J\alpha\beta}\right)\tilde{L}^q\partial_t^s\partial_\alpha\partial_\beta u_J,
\end{aligned} \tag{2.83}$$

and

$$\begin{aligned}
&\left(\partial_t^2 - c_I\Delta + a\partial_t\right)L^l Z^\nu u_I + \sum_{\substack{1\leqslant J\leqslant D\\ 0\leqslant\alpha,\beta\leqslant 3}} {\gamma_I}^{J\alpha\beta}\partial_\alpha\partial_\beta L^l Z^\nu u_J\\
&= (L+2)^l Z^\nu F_I + \sum_{\substack{p+q\leqslant l\\ |\kappa|+|\delta|=|\nu|\\ q+|\delta|\leqslant l+|\nu|-1}} C_{p,q,\kappa,\nu}\cdot\left(\tilde{L}^p Z^\kappa a\right)L^q Z^\delta\partial_t u_I\\
&\quad + \sum_{J=1}^{D}\sum_{\substack{p+q\leqslant l\\ |\kappa|+|\delta|=|\nu|\\ q+|\delta|\leqslant l+|\nu|-1\\ |\eta|=2}} C_{p,q,\kappa,\delta,\eta,J}\cdot\left(L^p Z^\kappa{\gamma_I}_J{}^{\alpha\beta}\right)L^q Z^\delta\partial^\eta u_J,
\end{aligned} \tag{2.84}$$

where $\chi_{k,\nu}$ and $\chi_{k,\nu,J}$ are smooth function dependent on lower indices which supports are in the region $\{x\in\mathbb{R}^3\backslash\mathcal{K}\colon\ \chi(x)\leqslant 2\}$, and $C_{p,q,\beta,\nu}$ and $C_{p,q,r,s,\eta,J}$ are constants dependent on the lower indices.

When we construct the energy estimates for the derivatives of the solution, we use the following estimates which follows from the elliptic regularity. For any $M\geqslant 0$ and $l\geqslant 0$, we have

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ l\leqslant\mu_0}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \leqslant C\sum_{\substack{\mu+j\leqslant M\\ l\leqslant\mu_0}} \left\|\left(\tilde{L}^\mu\partial_t^j u\right)'(t,\cdot)\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}$$
$$+C\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ l\leqslant\mu_0}} \left\|L^\mu\partial^\alpha(\Box+a\partial_t)u(t,\cdot)\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \tag{2.85}$$

for any function u which satisfies the Dirichlet condition $u|_{\partial\mathcal{K}}=0$, where C is a constant dependent of M, l_0, and a, but independent of u.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We put $\|\cdot\|_2 := \|\cdot\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}$ and $S_T := [0,T]\times(\mathbb{R}^3\setminus\mathcal{K})$.

Proposition 3.1. *Let M_0 be a sufficiently large number (for example, we can take $M_0=120$). Let $\varepsilon>0$ be a sufficiently small real number. Let $\mathcal{K}$, a, F, f and g satisfy the assumption in Theorem 1.1, and let u be the solution of (1.1). We assume*

$$\sum_{|\alpha|\leqslant 2M_0+20} \left\|\langle x\rangle^{|\alpha|}\partial_x^\alpha f(x)\right\|_{L_x^2} + \sum_{|\alpha|\leqslant 2M_0+19} \left\|\langle x\rangle^{|\alpha|+1}\partial_x^\alpha g(x)\right\|_{L_x^2} \leqslant \varepsilon. \tag{3.1}$$

If there exists a real number $A_0>0$ such that u satisfies

$$\sum_{|\alpha|\leqslant M_0} \sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}} \left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\big(1+t+|x|\big)\left|Z^\alpha u_I(t,x)\right| \leqslant A_0\varepsilon, \tag{3.2}$$
$$\sum_{|\alpha|\leqslant M_0-1} \sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}} \big(1+t+|x|\big)\left|Z^\alpha\partial u_I(t,x)\right| \leqslant A_0\varepsilon \tag{3.3}$$

for any $1\leqslant I\leqslant D$, then for any $\mu_0\geqslant 0$, $M\leqslant 2M_0-20-20\mu_0$ and sufficiently small $\sigma>0$, there exists a positive constant C_{M,μ_0} such that the inequality

$$\sum_{\substack{\mu+j\leqslant M\\ \mu\leqslant\mu_0}} \left\|\partial\tilde{L}^\mu\partial_t^j u\right\|_{L^\infty((0,T),L^2)}$$
$$+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu\partial^\alpha u'\right\|_{L^\infty((0,T),L^2)}$$
$$+\sum_{\substack{\mu+|\alpha|\leqslant M-2\\ \mu\leqslant\mu_0}} \big(\log(e+T)\big)^{-1/2}\left\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\right\|_{L^2((0,T)\times\mathbb{R}^3\setminus\mathcal{K})}$$
$$+\sum_{\substack{\mu+|\alpha|\leqslant M-3\\ \mu\leqslant\mu_0}} \left\|L^\mu Z^\alpha u'\right\|_{L^\infty((0,T),L^2)}$$
$$+\sum_{\substack{\mu+|\alpha|\leqslant M-5\\ \mu\leqslant\mu_0}} \big(\log(e+T)\big)^{-1/2}\left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\right\|_{L^2((0,T)\times\mathbb{R}^3\setminus\mathcal{K})}$$

$$+\sum_{\substack{\mu+|\alpha|\leqslant M-11\\ \mu\leqslant\mu_0}}\sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\left(1+\log\frac{1+t+|x|}{1+|c_It-|x||}\right)^{-1}\big(1+|x|\big)\left|L^\mu Z^\alpha u_I(t,x)\right|$$

$$+\sum_{\substack{\mu+|\alpha|\leqslant M-13\\ \mu\leqslant\mu_0-1}}\sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\left(1+\log\frac{1+t+|x|}{1+|c_It-|x||}\right)^{-1}\big(1+t+|x|\big)\left|L^\mu Z^\alpha u_I(t,x)\right|$$

$$\leqslant C_{M,\mu_0}\varepsilon(1+T)^{C_{M,\mu_0}(\varepsilon+\sigma)} \tag{3.4}$$

holds. Moreover, the following inequalities hold:

$$(1)\quad \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}}\sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\left(1+\log\frac{1+t+|x|}{1+|c_It-|x||}\right)^{-1}\big(1+t+|x|\big)\left|L^\mu Z^\alpha u_I(t,x)\right|$$

$$\leqslant C_0\sum_{\substack{j+\mu+|\alpha|\leqslant M+8\\ \mu\leqslant\mu_0+1\\ j\leqslant 1}}\left\|\big(\langle y\rangle\partial\big)^j L^\mu Z^\alpha u(0,y)\right\|_{L^2_y}$$

$$+C\sum_{\substack{\mu+|\alpha|\leqslant M+8\\ \mu\leqslant\mu_0}}\left\|\langle y\rangle^{-1/2}L^\mu Z^\alpha u'\right\|^2_{L^2((0,T)\times\mathbb{R}^3\setminus\mathcal{K})}$$

$$+C\varepsilon^3(1+T)^{C(\varepsilon+\sigma)}, \tag{3.5}$$

where C_0 is independent of A_0, C is dependent on C_{M+19,μ_0-1}.

$$(2)\quad \sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}}\langle x\rangle^{1/2+\theta}\big\langle c_It-|x|\big\rangle^{1-\theta}\left|L^\mu Z^\alpha u_I'(t,x)\right|\leqslant C\varepsilon(1+T)^{C(\varepsilon+\sigma)} \tag{3.6}$$

for any $0\leqslant\theta\leqslant 1/2$, where C is dependent on $C_{M+5,2}$ and $C_{M/2+15,2}$.

$$(3)\quad \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}}\left\|L^\mu Z^\alpha u'\right\|_{L^\infty((0,T),L^2)}\leqslant C_0\sum_{\substack{j+\mu+|\alpha|\leqslant M+10\\ \mu\leqslant 3\\ j\leqslant 1}}\left\|\big(\langle y\rangle\partial\big)^j L^\mu Z^\alpha u(0,y)\right\|_{L^2_y}+C\varepsilon^{3/2}, \tag{3.7}$$

where C_0 is independent of A_0, C is dependent on $C_{M+15,3}$ and $C_{M+21,1}$.

$$(4)\quad \sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\sum_{|\alpha|\leqslant M}\langle x\rangle^{1/2+\theta}\big\langle c_It-|x|\big\rangle^{1-\theta}\left|Z^\alpha u_I'(t,x)\right|$$

$$\leqslant C_0\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}}\left\|L^\mu Z^\alpha u'\right\|_{L^\infty((0,T),L^2)}+C_0\sum_{|\alpha\leqslant M+2}\left\|Z^\alpha u'\right\|^2_{L^\infty((0,T),L^2)}+C\varepsilon, \tag{3.8}$$

for any $0\leqslant\theta\leqslant 1/2$, where C_0 is independent of A_0, C is dependent on $C_{M+13,0}$.

$$(5)\quad \sum_{|\alpha|\leqslant M_0}\sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\big(1+t+|x|\big)\big|Z^\alpha u_I(t,x)\big|$$

$$\leqslant C_0\sum_{\substack{j+\mu+|\alpha|\leqslant M_0+18\\ \mu\leqslant 3\\ j\leqslant 1}}\big\|\big(\langle y\rangle\partial\big)^j L^\mu Z^\alpha u(0,y)\big\|_{L^2_y}+C\varepsilon^2, \tag{3.9}$$

where C_0 is independent of A_0, C is dependent on $C_{M_0+23,3}$ and $C_{M_0+29,1}$.

$$(6)\quad \sum_{|\alpha|\leqslant M_0-1}\sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\big(1+t+|x|\big)\big|Z^\alpha u'\big|$$

$$\leqslant C_0\sum_{\substack{j+\mu+|\alpha|\leqslant M_0+18\\ \mu\leqslant 3\\ j\leqslant 1}}\big\|\big(\langle y\rangle\partial\big)^j L^\mu Z^\alpha u(0,y)\big\|_{L^2_y}+C\varepsilon^{3/2}, \tag{3.10}$$

where C_0 is independent of A_0, C is dependent on $C_{M_0+23,3}$ and $C_{M_0+29,1}$.

3.1. Proof of Theorem 1.1

Here we prove Theorem 1.1. We use the continuity argument which shows that the local in time solution u does not blow up if its initial data is sufficiently small. We refer to [21,26] for the existence of the local in time solutions. By (5) and (6) of Proposition 3.1, there exists a universal constant C_0 which is independent of A_0 such that

$$\sum_{\alpha|\leqslant M_0}\sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\big(1+t+|x|\big)\big|Z^\alpha u_I(t,x)\big|$$

$$+\sum_{\alpha|\leqslant M_0-1}\sup_{\substack{(t,x)\in S_T\\ 1\leqslant I\leqslant D}}\big(1+t+|x|\big)\big|Z^\alpha\partial u_I(t,x)\big|\leqslant C_0\varepsilon+C\varepsilon^{3/2}, \tag{3.11}$$

where C is dependent on A_0 and $C_{M_0+23,3}$ and $C_{M_0+29,1}$. We put $A_0=4C_0$ and take ε sufficiently small such that $C\varepsilon^{3/2}\leqslant C_0\varepsilon$. Then the right hand side of (3.11) is bounded by $A_0\varepsilon/2$, which and (3.2), (3.3) show the local in time solution u does not blow up, namely the solution exists globally in time. □

3.2. Basic estimates for the nonlinearities

Before we start the proof of Proposition 3.1, we show several estimates which show how we treat the nonlinearities. We drop the lower index I of u_I when it is not essential in our estimates. And we denote u or ∂u by $\bar{\partial}u$. For a real number s, $s-$ denotes a number which is strictly smaller than s and close to s.

First we consider the semilinear part of quadratic nonlinearities. Since we have

$$\begin{aligned}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left|L^\mu\partial^\alpha\left(u'u'\right)\right| \lesssim{}& \sum_{|\alpha|\leqslant M_0-1} \left|\partial^\alpha u'\right| \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left|L^\mu\partial^\alpha u'\right| + \sum_{M_0\leqslant|\alpha|\leqslant M-M_0} \left|\partial^\alpha u'\right|^2\\ &+ \sum_{M_0\leqslant|\alpha|\leqslant M-1} \left|\partial^\alpha u'\right| \sum_{\substack{\mu+|\alpha|\leqslant M-M_0\\ 1\leqslant\mu\leqslant\mu_0}} \left|L^\mu\partial^\alpha u'\right|\\ &+ \sum_{\substack{\mu+|\alpha|\leqslant M/2\\ 1\leqslant\mu\leqslant\mu_0-1}} \left|L^\mu\partial^\alpha u'\right| \sum_{\substack{\mu+|\alpha|\leqslant M-1\\ 1\leqslant\mu\leqslant\mu_0-1}} \left|L^\mu\partial^\alpha u'\right|, \end{aligned} \tag{3.12}$$

so that, we obtain by the Sobolev estimate Lemma 2.17 and the assumption (3.3)

$$\begin{aligned}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu\partial^\alpha\left(u'u'\right)\right\|_2 \lesssim{}& \frac{\varepsilon}{1+t} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu\partial^\alpha u'\right\|_2\\ &+ \sum_{M_0\leqslant|\alpha|\leqslant M-M_0} \left\|\langle x\rangle^{-1/2}\partial^\alpha u'\right\|_2 \sum_{M_0\leqslant|\alpha|\leqslant M-M_0+2} \left\|\langle x\rangle^{-1/2}Z^\alpha u'\right\|_2\\ &+ \sum_{M_0\leqslant|\alpha|\leqslant M-1} \left\|\langle x\rangle^{-1/2}\partial^\alpha u'\right\|_2 \sum_{\substack{\mu+|\alpha|\leqslant M-M_0+2\\ 1\leqslant\mu\leqslant\mu_0}} \left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\right\|_2\\ &+ \sum_{\substack{\mu+|\alpha|\leqslant M-1\\ 1\leqslant\mu\leqslant\mu_0-1}} \left\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\right\|_2 \sum_{\substack{|\alpha|\leqslant M/2+2\\ 1\leqslant\mu\leqslant\mu_0-1}} \left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\right\|_2. \end{aligned} \tag{3.13}$$

Similarly, for the quasilinear part of the nonlinearities, we obtain

$$\begin{aligned}&\sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant\mu_0\\ \nu+|\beta|\leqslant M-1}} \left\|\left(L^\mu\partial^\alpha u'\right)\left(L^\nu\partial^\beta u''\right)\right\|_2\\ &\lesssim \frac{\varepsilon}{1+t} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu\partial^\alpha u'\right\|_2\\ &\quad+ \sum_{M_0\leqslant|\alpha|\leqslant M-M_0+1} \left\|\langle x\rangle^{-1/2}\partial^\alpha u'\right\|_2 \sum_{M_0\leqslant|\alpha|\leqslant M-M_0+3} \left\|\langle x\rangle^{-1/2}Z^\alpha u'\right\|_2\\ &\quad+ \sum_{M_0\leqslant|\alpha|\leqslant M} \left\|\langle x\rangle^{-1/2}\partial^\alpha u'\right\|_2 \sum_{\substack{\mu+|\alpha|\leqslant M-M_0+3\\ 1\leqslant\mu\leqslant\mu_0}} \left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\right\|_2\\ &\quad+ \sum_{\substack{\mu+|\alpha|\leqslant M\\ 1\leqslant\mu\leqslant\mu_0-1}} \left\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\right\|_2 \sum_{\substack{|\alpha|\leqslant M/2+3\\ 1\leqslant\mu\leqslant\mu_0-1}} \left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\right\|_2. \end{aligned} \tag{3.14}$$

Second, for the semilinear part of the cubic nonlinearities, since we have by the assumption (3.2)

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \big|L^\mu Z^\alpha(\bar\partial u\bar\partial u\bar\partial u)\big| &\lesssim \sum_{|\alpha|\leqslant M_0}\big|Z^\alpha u\big|^2\Bigg(|u|+\sum_{\substack{1\leqslant\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big|L^\mu Z^\alpha u\big|+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big|L^\mu Z^\alpha u'\big|\Bigg)\\
&\quad+\sum_{|\alpha|\leqslant M_0}\big|Z^\alpha u\big|\sum_{M_0\leqslant|\alpha|\leqslant M}\big|Z^\alpha u\big|\sum_{\substack{\mu+|\alpha|\leqslant M-M_0+1\\ 1\leqslant\mu\leqslant\mu_0}}\big|L^\mu Z^\alpha u\big|\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-2}}\big|L^\mu Z^\alpha u\big|\sum_{\substack{\mu+|\alpha|\leqslant M/2+1\\ \mu\leqslant\mu_0-1}}\big|L^\mu Z^\alpha u\big|\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}}\big|L^\mu Z^\alpha u\big|\\
&\lesssim\left(\frac{\varepsilon\log(2+t)}{1+t+|x|}\right)^3+\frac{(\varepsilon\log(2+t))^2}{1+t+|x|}\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}}\big|L^\mu Z^\alpha u'\big|\\
&\quad+\left(\frac{\varepsilon\log(2+t)}{1+t+|x|}\right)^2\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big|L^\mu Z^\alpha u'\big|\\
&\quad+\frac{(\varepsilon\log(2+t))^3}{(1+t+|x|)(1+|x|)^2}(1+t)^{C(\varepsilon+\sigma)},
\end{aligned}\tag{3.15}$$

where C is dependent on C_{M+13,μ_0-1} and C_{M-M_0+12,μ_0}, and we can remove the last term when $\mu_0=0$. So that, we obtain

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big\|L^\mu Z^\alpha(\bar\partial u\bar\partial u\bar\partial u)\big\|_2 &\lesssim\frac{\varepsilon^3}{(1+t)^{3/2-}}+\frac{(\varepsilon\log(2+t))^2}{1+t}\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}}\big\|L^\mu Z^\alpha u'\big\|_2\\
&\quad+\frac{\varepsilon^2}{(1+t)^{2-}}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big\|L^\mu Z^\alpha u'\big\|_2\\
&\quad+\frac{(\varepsilon\log(2+t))^3}{1+t}(1+t)^{C(\varepsilon+\sigma)},
\end{aligned}\tag{3.16}$$

where C is dependent on C_{M+13,μ_0-1} and C_{M-M_0+12,μ_0}, and we can remove the last term when $\mu_0=0$. Similarly, we obtain for the quasilinear part of the cubic nonlinearities

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant\mu_0\\ \nu+|\beta|\leqslant M-1}}\big\|\big(L^\mu Z^\alpha(\bar\partial u\bar\partial u)\big)\big(L^\nu Z^\beta\partial^2 u\big)\big\|_2\lesssim\frac{\varepsilon^3}{(1+t)^{3/2-}}\\
&\quad+\frac{(\varepsilon\log(2+t))^2}{1+t}\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}}\big\|L^\mu Z^\alpha u'\big\|_2+\frac{\varepsilon^2}{(1+t)^{2-}}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big\|L^\mu Z^\alpha u'\big\|_2\\
&\quad+\frac{(\varepsilon\log(2+t))^3}{1+t}(1+t)^{C(\varepsilon+\sigma)},
\end{aligned}\tag{3.17}$$

where C is dependent on C_{M+14,μ_0-1} and C_{M-M_0+13,μ_0}, and we can remove the last term when $\mu_0=0$.

3.3. Proof of Proposition 3.1

First we show (3.4).

3.3.1. The estimate for $\|L^\mu\partial^\alpha u'\|_2$

We start from the estimate (2.85)

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u'\right\|_2\lesssim\sum_{\substack{\mu+j\leqslant M\\ \mu\leqslant\mu_0}}\left\|\partial\tilde{L}^\mu\partial_t^j u\right\|_2+\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha(\Box+a\partial_t)u\right\|_2,\tag{3.18}$$

and we estimate the last term by the analogous argument to derive (3.13) and (3.16)

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha(\Box+a\partial_t)u\right\|_2\\
&\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\{\left\|L^\mu\partial^\alpha\left(u'u'\right)\right\|_2+\left\|L^\mu\partial^\alpha(\bar\partial u\bar\partial u\bar\partial u)\right\|_2\right\}\\
&\lesssim\frac{\varepsilon}{1+t}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u'\right\|_2+\sum_{M_0\leqslant|\alpha|\leqslant M-M_0}\left\|\partial^\alpha u'\right\|_2\sum_{M_0\leqslant|\alpha|\leqslant M-M_0}\left\|\partial^\alpha u'\right\|_2\\
&\quad+\sum_{M_0\leqslant|\alpha|\leqslant M-1}\left\|\partial^\alpha u'\right\|_2\sum_{\substack{M_0\leqslant\mu+|\alpha|\leqslant M-M_0\\ 1\leqslant\mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u'\right\|_2\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ 1\leqslant\mu\leqslant\mu_0-1}}\left\|L^\mu\partial^\alpha u'\right\|_2\sum_{\substack{\mu+|\alpha|\leqslant M/2\\ 1\leqslant\mu\leqslant\mu_0-1}}\left\|L^\mu\partial^\alpha u'\right\|_2\\
&\quad+\frac{\varepsilon^3}{(1+t)^{3/2-}}+\frac{(\varepsilon\log(2+t))^2}{1+t}\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}}\left\|L^\mu\partial^\alpha u'\right\|_2\\
&\quad+\frac{\varepsilon^2}{(1+t)^{2-}}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u'\right\|_2+\frac{(\varepsilon\log(2+t))^3}{1+t}(1+t)^{C(\varepsilon+\sigma)},
\end{aligned}\tag{3.19}$$

where C is dependent on C_{M+13,μ_0-1} and C_{M-M_0+12,μ_0}, and we can remove the last term when $\mu_0=0$. So that, by induction argument on (3.4), we obtain for sufficiently small $\varepsilon>0$

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u'\right\|_2\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|\partial\tilde{L}^\mu\partial^\alpha u\right\|_2+\varepsilon(1+t)^{C(\varepsilon+\sigma)},\tag{3.20}$$

where C is dependent on C_{M+13,μ_0-1} and C_{M-M_0+12,μ_0}.

3.3.2. The estimate for the boundary term

To consider the estimate for $\|\partial\tilde{L}^\mu\partial_t^j u\|_2$ in the next subsection, we prepare the estimate for the boundary term. By Lemma 2.2, we have

$$
\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}}\int_0^t \left\|L^\mu\partial^\alpha u'\right\|_{L^2(|x|<2)}\,ds\\
&\lesssim \varepsilon+\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0-1}}\int_0^t\int_0^s \left\|L^\mu\partial^\alpha(\Box+a\partial_t)u\right\|_{L^2(||x|-(s-\tau)|<4)}\,d\tau\,ds\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0-1}}\int_0^t \left\|L^\mu\partial^\alpha(\Box+a\partial_t)u\right\|_{L^2(|x|<4)}\,ds\\
&\lesssim \varepsilon+\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0-1}}\int_0^t\int_0^s \left\{\left\|L^\mu\partial^\alpha\big(u'u'\big)\right\|_{L^2(||x|-(s-\tau)|<4)}+\left\|L^\mu\partial^\alpha(\bar\partial u\bar\partial u\bar\partial u)\right\|_{L^2(||x|-(s-\tau)|<4)}\right\}d\tau\,ds\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0-1}}\int_0^t \left\{\left\|L^\mu\partial^\alpha\big(u'u'\big)\right\|_{L^2(|x|<4)}\,ds+\left\|L^\mu\partial^\alpha(\bar\partial u\bar\partial u\bar\partial u)\right\|_{L^2(|x|<4)}\right\}ds\\
&=: \varepsilon+B_1+B_2,
\end{aligned}
\tag{3.21}
$$

where we have put the last two terms in the last right hand side as B_1 and B_2. We use the bound

$$
\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0-1}}\left|L^\mu\partial^\alpha\big(u'u'\big)\right|\lesssim\sum_{\substack{\mu+|\alpha|\leqslant (M+2)/2\\ \mu\leqslant\mu_0-1}}\left|L^\mu\partial^\alpha u'\right|\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0-1}}\left|L^\mu\partial^\alpha u'\right|
\tag{3.22}
$$

and

$$
\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0-1}}\left|L^\mu\partial^\alpha(\bar\partial u\bar\partial u\bar\partial u)\right|&\lesssim\Big(\sum_{|\alpha|\leqslant(M+4)/2}\left|\partial^\alpha u\right|\Big)^2\sum_{|\alpha|\leqslant M+3}\left|\partial^\alpha u\right|\\
&\quad+\sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0-2}}\left|L^\mu\partial^\alpha u\right|\sum_{\substack{\mu+|\alpha|\leqslant (M+4)/2\\ \mu\leqslant\mu_0-1}}\left|L^\mu\partial^\alpha u\right|\sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0-1}}\left|L^\mu\partial^\alpha u\right|\\
&\lesssim\frac{(\varepsilon\log(2+t))^3}{(1+\tau+|y|)(1+|y|)^2}(1+\tau)^{C(\varepsilon+\sigma)},
\end{aligned}
\tag{3.23}
$$

where C is dependent on C_{M+16,μ_0-1}. So that, by Lemma 2.17, we have

$$
B_2\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0-1}}\left\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\right\|^2_{L^2_{t,x}}+\varepsilon^3(1+t)^{C(\varepsilon+\sigma)}\lesssim\varepsilon^2(1+t)^{C(\varepsilon+\sigma)},
\tag{3.24}
$$

where C is dependent on C_{M+16,μ_0-1}.

For the term B_1, we use the estimates

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0-1}} \left\| L^\mu \partial^\alpha \left(u'u'\right)\right\|_{L^2(|y|-(s-\tau)<4)} \\
&\lesssim \sum_{\substack{\mu+|\alpha|\leqslant (M+6)/2\\ \mu\leqslant\mu_0-1}} \left\| \langle y\rangle^{-1/2} L^\mu Z^\alpha u'\right\|_{L^2(||y|-(s-\tau)|<5)} \sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0-1}} \left\| \langle y\rangle^{-1/2} L^\mu \partial^\alpha u'\right\|_{L^2(||y|-(s-\tau)|<4)} \\
&\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0-1}} \left\| \langle y\rangle^{-1/2} L^\mu Z^\alpha u'\right\|^2_{L^2(||y|-(s-\tau)|<5)},
\end{aligned} \tag{3.25}$$

which follows from Lemma 2.17, and

$$\left\| \frac{(\varepsilon \log(2+t))^3}{(1+s)(1+s-\tau)^2}(1+\tau)^{C(\varepsilon+\sigma)}\right\|_{L^2(||y|-(s-\tau)|<4)} \lesssim \frac{(\varepsilon \log(2+t))^3}{(1+s)(1+s-\tau)}(1+\tau)^{C(\varepsilon+\sigma)} \tag{3.26}$$

to obtain

$$B_1 \lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0-1}} \left\| \langle y\rangle^{-1/2} L^\mu Z^\alpha u'\right\|^2_{L^2_{t,x}} + \varepsilon^3 (1+t)^{C(\varepsilon+\sigma)}, \tag{3.27}$$

where C is dependent on C_{M+16,μ_0-1}.

Finally, we obtain the estimate

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}} \int_0^t \left\| L^\mu \partial^\alpha u'\right\|_{L^2(|x|<2)}\, ds \lesssim \varepsilon (1+t)^{C(\varepsilon+\sigma)}, \tag{3.28}$$

where C is dependent on C_{M+16,μ_0-1}.

3.3.3. The estimate for $\|\partial \tilde{L}^\mu \partial_t^j u\|_2$

Since $\tilde{L}^\mu \partial_t^j u$ satisfies the Dirichlet condition, by the energy estimate (2.2) we have

$$\partial_t \left\{ \int_{\mathbb{R}^3\setminus\mathcal{K}} e_0\left(\tilde{L}^\mu \partial_t^j u\right) dx \right\}^{1/2} \lesssim \left\| (\Box_\gamma + a\partial_t)\tilde{L}^\mu \partial_t^j u\right\|_2 + \left\|\gamma'\right\|_\infty \left\{ \int_{\mathbb{R}^3\setminus\mathcal{K}} e_0\left(\tilde{L}^\mu \partial_t^j u\right) dx \right\}^{1/2}, \tag{3.29}$$

where we put

$$\gamma_I{}^{Kkl} := \sum_{0\leqslant j\leqslant 3} Q_I^{JKjkl} \partial_j u_J + R_I^{Kkl}\left(u, u'\right), \tag{3.30}$$

$$\Box_{\gamma_I} u_I := \left(\partial_t^2 - c_I^2 \Delta\right) u_I - \sum_{\substack{1\leqslant K\leqslant D\\ 0\leqslant k,l\leqslant 3}} \gamma_I{}^{Kkl} \partial_k \partial_l u_K \tag{3.31}$$

and we drop I of u_I and γ_I in the following. We write the first term in the right hand side as

$$
\begin{aligned}
(\Box_\gamma + a\partial_t)\tilde{L}^\mu \partial_t^j u = \tilde{L}^\mu \partial_t^j (\Box_\gamma + a\partial_t)u + \big[\Box, L^\mu \partial_t^j\big]u \\
+ \big[\Box, \big(\tilde{L}^\mu - L^\mu\big)\partial_t^j\big]u + \big[\gamma\partial^2 + a\partial_t, \tilde{L}^\mu \partial_t^j\big]u
\end{aligned}
\tag{3.32}
$$

and the each term of the right hand side can be bounded by the estimates in Section 2.6 as

$$
\big|\tilde{L}^\mu \partial_t^j (\Box_\gamma + a\partial_t)u\big| \lesssim \sum_{\substack{\nu+|\alpha|\leqslant \mu+j \\ \nu\leqslant\mu}} \big|L^\nu \partial^\alpha (\Box_\gamma + a\partial_t)u\big|, \tag{3.33}
$$

$$
\begin{aligned}
&\big|\big[\Box, L^\mu \partial_t^j\big]u\big| \\
&\quad\lesssim \sum_{\substack{\nu+|\alpha|\leqslant \mu-1+j \\ \nu\leqslant\mu-1}} \big|L^\nu \partial^\alpha (\Box_\gamma + a\partial_t)u\big| \\
&\qquad + \sum_{\substack{\nu_1+|\alpha_1|+\nu_2+|\alpha_2|\leqslant \mu-1+j \\ \nu_1+\nu_2\leqslant\mu-1}} \big\{\big|L^{\nu_1}\partial^{\alpha_1}\gamma\big| \cdot \big|L^{\nu_2}\partial^{\alpha_2}u''\big| + \big|L^{\nu_1}\partial^{\alpha_1}a\big| \cdot \big|L^{\nu_2}\partial^{\alpha_2}\partial_t u\big|\big\},
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&\big|\big[\Box, \big(\tilde{L}^\mu - L^\mu\big)\partial_t^j\big]u\big| \\
&\quad\lesssim \chi_{\{|x|\leqslant 2\}} \sum_{\substack{\nu+|\alpha|\leqslant \mu+j \\ \nu\leqslant\mu-1}} \big|L^\nu \partial^\alpha u'\big| + \sum_{\substack{\nu+|\alpha|\leqslant \mu+j-1 \\ \nu\leqslant\mu-2}} \big|L^\nu \partial^\alpha (\Box_\gamma + a\partial_t)u\big| \\
&\qquad + \sum_{\substack{\nu_1+|\alpha_1|+\nu_2+|\alpha_2|\leqslant \mu+j-1 \\ \nu_1+\nu_2\leqslant\mu-2}} \big\{\big|L^{\nu_1}\partial^{\alpha_1}\gamma\big| \cdot \big|L^{\nu_2}\partial^{\alpha_2}u''\big| + \big|L^{\nu_1}\partial^{\alpha_1}a\big| \cdot \big|L^{\nu_2}\partial^{\alpha_2}\partial_t u\big|\big\},
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
\big|\big[\gamma\partial^2 + a\partial_t, \tilde{L}^\mu \partial_t^j\big]u\big| \lesssim{}& \chi_{\{|x|\leqslant 2\}} \sum_{\substack{\nu+|\alpha|\leqslant \mu+j \\ \nu\leqslant\mu-1}} \big|\gamma L^\nu \partial^\alpha u\big| + \chi_{\{|x|\leqslant 2\}} \sum_{\substack{\nu+|\alpha|\leqslant \mu+j \\ \nu\leqslant\mu-1}} \big|\gamma L^\nu \partial^\alpha u'\big| \\
&+ \sum_{\substack{\nu_1+|\alpha_1|+\nu_2+|\alpha_2|\leqslant \mu+j \\ \nu_1+\nu_2\leqslant\mu \\ \nu_2+\alpha_2\leqslant\mu+j-1}} \big|L^{\nu_1}\partial^{\alpha_1}\gamma\big| \cdot \big|L^{\nu_2}\partial^{\alpha_2}u''\big| \\
&+ \sum_{\substack{\nu+|\alpha|\leqslant \mu+j \\ \nu\leqslant\mu-1}} \big|aL^\nu \partial^\alpha u\big| + \sum_{\substack{\nu+|\alpha|\leqslant \mu+j \\ \nu\leqslant\mu-1}} \big|aL^\nu \partial^\alpha u'\big|,
\end{aligned}
\tag{3.36}
$$

where $\chi_{\{|x|\leqslant 2\}}$ denotes the smooth function with the compact support in $\{x \in \mathbb{R}^3\backslash\mathcal{K}\colon\ |x| \leqslant 2\}$. So that, we have

$$
\begin{aligned}
\sum_{\substack{\mu+j\leqslant M \\ \mu\leqslant\mu_0}} \partial_t \bigg\{ \int\limits_{\mathbb{R}^3\backslash\mathcal{K}} e_0\big(\tilde{L}^\mu \partial_t^j u\big)\,dx \bigg\}^{1/2} \lesssim{}& \|\gamma'\|_\infty \bigg\{ \int\limits_{\mathbb{R}^3\backslash\mathcal{K}} e_0\big(\tilde{L}^\mu \partial_t^j u\big)\,dx \bigg\}^{1/2} \\
&+ \big(1 + \|\gamma\|_\infty\big) \sum_{\substack{\mu+|\alpha|\leqslant M \\ \mu\leqslant\mu_0-1}} \big\|L^\mu \partial^\alpha u'\big\|_{L^2(|x|<2)}
\end{aligned}
$$

$$+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha(\Box_\gamma+a\partial_t)u\right\|_2$$
$$+\sum_{\substack{\mu_1+|\alpha_1|+\mu_2+|\alpha_2|\leqslant M\\ \nu_1+\nu_2\leqslant\mu_0\\ \mu_2+|\alpha_2|\leqslant M-1}}\left\|\left(L^{\mu_1}\partial^{\alpha_1}\gamma\right)\cdot\left(L^{\nu_2}\partial^{\alpha_2}u''\right)\right\|_2. \tag{3.37}$$

Using the estimates (3.13) and (3.17), the last two terms are bounded by

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha(\Box_\gamma+a\partial_t)u\right\|_2+\sum_{\substack{\mu_1+|\alpha_1|+\mu_2+|\alpha_2|\leqslant M\\ \nu_1+\nu_2\leqslant\mu_0\\ \mu_2+|\alpha_2|\leqslant M-1}}\left\|\left(L^{\mu_1}\partial^{\alpha_1}\gamma\right)\cdot\left(L^{\nu_2}\partial^{\alpha_2}u''\right)\right\|_2$$
$$\lesssim\frac{\varepsilon}{1+t}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\left\|L^\mu\partial^\alpha u'\right\|_2+B, \tag{3.38}$$

where

$$\begin{aligned}B:={}&\sum_{M_0\leqslant|\alpha|\leqslant M-M_0+1}\left\|\langle x\rangle^{-1/2}\partial^\alpha u'\right\|_2\sum_{M_0\leqslant|\alpha|\leqslant M-M_0+3}\left\|\langle x\rangle^{-1/2}Z^\alpha u'\right\|_2\\
&+\sum_{M_0\leqslant|\alpha|\leqslant M}\left\|\langle x\rangle^{-1/2}\partial^\alpha u'\right\|_2\sum_{\substack{\mu+|\alpha|\leqslant M-M_0+3\\ 1\leqslant\mu\leqslant\mu_0}}\left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\right\|_2\\
&+\sum_{\substack{\mu+|\alpha|\leqslant M\\ 1\leqslant\mu\leqslant\mu_0-1}}\left\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\right\|_2\sum_{\substack{\mu+|\alpha|\leqslant M/2+3\\ 1\leqslant\mu\leqslant\mu_0-1}}\left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\right\|_2\\
&+\frac{(\varepsilon\log(2+t))^2}{1+t}\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}}\left\|L^\mu\partial^\alpha u'\right\|_2\\
&+\varepsilon^3(1+t)^{-3/2+}+\frac{(\varepsilon\log(2+t))^3}{1+t}(1+t)^{C(\varepsilon+\sigma)},\end{aligned} \tag{3.39}$$

where the constant C is dependent on C_{M+14,μ_0-1} and C_{M-M_0+13,μ_0} and we can remove the last term when $\mu_0=0$. Using the Gronwall inequality, we have

$$\sum_{\substack{\mu+j\leqslant M\\ \mu\leqslant\mu_0}}\left\{\int_{\mathbb{R}^3\setminus\mathcal{K}}e_0\left(\tilde{L}^\mu\partial_t^j u\right)dx\right\}^{1/2}$$
$$\lesssim\left[\varepsilon+\varepsilon^2(1+t)^{C(\varepsilon+\sigma)}+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}}\int_0^t\left\|L^\mu\partial^\alpha u'\right\|_{L^2(|x|<2)}ds+\int_0^t Bds\right](1+t)^{C\varepsilon}, \tag{3.40}$$

where C is dependent on C_{M-M_0+12,μ_0} and C_{M+13,μ_0-1}. Since we have (3.28) and

$$\int_0^t B\,ds \lesssim \varepsilon(1+t)^{C(\varepsilon+\sigma)} + \varepsilon^2\bigl(\log(2+t)\bigr)^3 \sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}} \bigl\|L^\mu \partial^\alpha u'\bigr\|_2, \tag{3.41}$$

where C is dependent on C_{M+14,μ_0-1} and C_{M-M_0+13,μ_0}, we obtain

$$\begin{aligned} &\sum_{\substack{\mu+j\leqslant M\\ \mu\leqslant\mu_0}} \biggl\{\int_{\mathbb{R}^3\setminus\mathcal{K}} e_0\bigl(\tilde{L}^\mu \partial_t^j u\bigr)\,dx\biggr\}^{1/2} \\ &\quad\lesssim \varepsilon(1+t)^{C(\varepsilon+\sigma)}\biggl(1+\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}} \bigl\|L^\mu\partial^\alpha u'\bigr\|_2\biggr), \end{aligned} \tag{3.42}$$

where C is dependent on C_{M+16,μ_0-1} and C_{M-M_0+13,μ_0}. By the use of the induction argument, we obtain

$$\sum_{\substack{\mu+j\leqslant M\\ j\leqslant\mu_0}} \bigl\|\partial\tilde{L}^\mu\partial_t^j u(t)\bigr\|_2 \leqslant C(1+t)^{C(\varepsilon+\sigma)} \tag{3.43}$$

for some positive constant C which is denoted by C_{M,μ_0}.

3.3.4. The estimate for $\|L^\mu Z^\alpha u'\|_2$

By the energy estimate (2.2) for $L^\mu Z^\alpha u$, we have

$$\begin{aligned} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \partial_t\int e_0\bigl(L^\mu Z^\alpha u\bigr)\,dx &\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \bigl\|L^\mu\partial^\alpha u'\bigr\|^2_{L^2(|x|<2)} \\ &\quad+ \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \biggl|\int \partial_t L^\mu Z^\alpha u(\Box_\gamma + a\partial_t)L^\mu Z^\alpha u\,dx\biggr| \\ &\quad+ \bigl\|\gamma'\bigr\|_\infty \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \bigl\|L^\mu Z^\alpha u'\bigr\|_2^2, \end{aligned} \tag{3.44}$$

where we have used the trace theorem for the boundary term, so that, there is a loss of one derivative. We estimate the second term in the right hand side by

$$\begin{aligned} &\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \biggl|\int \partial_t L^\mu Z^\alpha u(\Box_\gamma + a\partial_t)L^\mu Z^\alpha u\,dx\biggr| \\ &\quad\lesssim \int \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \bigl|\partial_t L^\mu Z^\alpha u\bigr| \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \bigl|(\Box_\gamma + a\partial_t)L^\mu Z^\alpha u\bigr|\,dx \end{aligned} \tag{3.45}$$

and we use the commutation estimates in Section 2.6 to obtain

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left|(\Box_\gamma + a\partial_t)L^\mu Z^\alpha u\right| \lesssim{} & \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left|L^\mu Z^\alpha(\Box_\gamma + a\partial_t)u\right| \\
& + \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant\mu_0\\ \nu+\beta\leqslant M-1}} \left|\bigl(L^\mu Z^\alpha \gamma\bigr)\bigl(L^\nu Z^\beta \partial^2 u\bigr)\right| \\
& + \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant\mu_0\\ \nu+\beta\leqslant M-1}} \left|\bigl(L^\mu Z^\alpha a\bigr)\bigl(L^\nu Z^\beta \partial_t u\bigr)\right| \qquad (3.46)
\end{aligned}$$

and then we obtain the bound

$$\begin{aligned}
& \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left|\int \partial_t L^\mu Z^\alpha u (\Box_\gamma + a\partial_t)L^\mu Z^\alpha u\, dx\right| \\
& \lesssim \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu Z^\alpha u'\right\|_2 \Biggl\{ \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu Z^\alpha(\Box_\gamma + a\partial_t)u\right\|_2 \\
& \quad + \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant\mu_0\\ \nu+\beta\leqslant M-1}} \left\|\bigl(L^\mu Z^\alpha \gamma\bigr)\bigl(L^\nu Z^\beta \partial^2 u\bigr)\right\|_2 \Biggr\} + \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha u'\right\|^2_{L^2(|x|<1)} \qquad (3.47)
\end{aligned}$$

where we have used $\operatorname{supp} a \subset \{|x| < 1\}$ to obtain the last term. Here, we apply the estimates in Section 3.2 to obtain

$$\begin{aligned}
& \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\mu Z^\alpha(\Box_\gamma + a\partial_t)u\right\|_2 + \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant\mu_0\\ \nu+\beta\leqslant M-1}} \left\|\bigl(L^\mu Z^\alpha \gamma\bigr)\bigl(L^\nu Z^\beta \partial^2 u\bigr)\right\|_2 \\
& \lesssim \frac{\varepsilon}{1+t} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left\|L^\nu Z^\alpha u'\right\|_2 + \tilde{B}, \qquad (3.48)
\end{aligned}$$

where $\tilde{B}$ is B in (3.39) with ∂ replaced by Z. Since $\|L^\mu Z^\alpha u'\|_2$ is equivalent to $e_0(L^\mu Z^\alpha u)$ by $\|\gamma'\|_\infty \lesssim \varepsilon/(1+t)$, we have

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \partial_t \int e_0\bigl(L^\mu Z^\alpha u\bigr)\, dx \lesssim{} & \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha u'\right\|^2_{L^2(|x|<2)} \\
& + \frac{\varepsilon}{1+t} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int e_0\bigl(L^\mu Z^\alpha u\bigr)\, dx \\
& + \Biggl(\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int e_0\bigl(L^\mu Z^\alpha u\bigr)\, dx \Biggr)^{1/2} \cdot \tilde{B}. \qquad (3.49)
\end{aligned}$$

So that, by the Gronwall inequality, we obtain

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \int e_0\bigl(L^\mu Z^\alpha u\bigr)\,dx\\
&\quad\lesssim \varepsilon\Biggl(1+\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\max\{\mu_0-1,0\}}} \bigl\|L^\mu Z^\alpha u'\bigr\|_2+\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \bigl\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\bigr\|_{L^2_{t,x}}\Biggr)(1+t)^{C(\varepsilon+\sigma)}, \qquad (3.50)
\end{aligned}$$

where C is dependent on C_{M-M_0+13,μ_0} and C_{M+14,μ_0-1}. With M replaced by $M-3$ and the induction argument, we obtain

$$\sum_{\substack{\mu+|\alpha|\leqslant M-3\\ \mu\leqslant\mu_0}} \bigl\|L^\mu Z^\alpha u'\bigr\|_2 \lesssim \varepsilon(1+t)^{C(\varepsilon+\sigma)} \qquad (3.51)$$

by the equivalence of $\|L^\mu Z^\alpha u'\|_2$ and $e_0(L^\mu Z^\alpha u)$.

3.3.5. The estimates for $\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\|_{L^2_{t,x}}$ *and* $\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\|_{L^2_{t,x}}$

By the weighted energy estimates Lemma 2.3, we have

$$\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \bigl(\log(2+t)\bigr)^{-1/2}\bigl\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\bigr\|_{L^2_{t,x}} \lesssim \varepsilon+\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant\mu_0}} \int_0^t \bigl\|L^\mu\partial^\alpha(\Box+a\partial_t)u\bigr\|_2\,ds, \qquad (3.52)$$

where the last term is bounded by

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0}} \int_0^t \bigl\{\bigl\|L^\mu\partial^\alpha\bigl(u'u'\bigr)\bigr\|_2+\bigl\|L^\mu\partial^\alpha(\bar\partial u\bar\partial u\bar\partial u)\bigr\|_2\bigr\}\,ds\\
&\quad\lesssim \varepsilon\log(2+t)^3\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant\mu_0}} \bigl\|L^\mu\partial^\alpha u'\bigr\|_2+C\varepsilon(1+t)^{C(\varepsilon+\sigma)}, \qquad (3.53)
\end{aligned}$$

where we have used the similar estimates in Section 3.2 for the last inequality, and C is dependent on C_{M+15,μ_0-1} and C_{M+14-M_0,μ_0}. We note that in the above estimates we can replace ∂ with Z. And replacing M with $M-2$ for ∂, M with $M-5$ for Z, we obtain the required estimates.

3.3.6. The estimate for $(1+\log\frac{1+t+|x|}{1+|c_It-|x||})^{-1}(1+|x|)|L^\mu Z^\alpha u_I(t,x)|$

By Proposition 2.9, we have

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \biggl(1+\log\frac{1+t+|x|}{1+|c_It-|x||}\biggr)^{-1}\bigl(1+|x|\bigr)\bigl|L^\mu Z^\alpha u_I(t,x)\bigr|\\
&\quad\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+6\\ \mu\leqslant\mu_0}}\sum_{j\leqslant 1} \bigl\|\bigl(\langle y\rangle\partial\bigr)^j L^\mu Z^\alpha u(0,y)\bigr\|_{L^2_y}+\sum_{\substack{\mu+|\alpha|\leqslant M+5\\ \mu\leqslant\mu_0}} \int_0^t\int_{\mathbb{R}^3\setminus\mathcal{K}} \bigl|L^\mu Z^\alpha(B+Q)\bigr|\frac{dy}{|y|}\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\sup_{(s,y)\in D_I(t,|x|)}|y|\big(1+|y|\big)\big(1+s+|y|\big)^{\sigma}z_\sigma\big(s,|y|\big)\big|L^\mu Z^\alpha(R+P)(s,y)\big|\\
&+\sum_{\substack{\mu+|\alpha|\leqslant M+4\\ \mu\leqslant\mu_0}}\sup_{(s,y)\in(0,t)\times\mathbb{R}^3\setminus\mathcal{K}}|y|\big(1+|y|\big)\big(1+s+|y|\big)^{\sigma}z_\sigma\big(s,|y|\big)\big|L^\mu\partial^\alpha(R+P)(s,y)\big|\\
=:&\sum_{\substack{\mu+|\alpha|\leqslant M+6\\ \mu\leqslant\mu_0}}\sum_{j\leqslant 1}\big\|\big(\langle y\rangle\partial\big)^jL^\mu Z^\alpha u(0)\big\|_2+B_1+B_2+B_3,
\end{aligned}
\tag{3.54}
$$

where we denote the last three terms by B_1, B_2, and B_3. The term B_1 is bounded by

$$
B_1\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M+6\\ \mu\leqslant\mu_0}}\big\|\langle y\rangle^{-1/2}L^\mu Z^\alpha u'\big\|_{L^2_{t,x}}.
\tag{3.55}
$$

To bound B_2, we put

$$
W(s,y)=W\big(s,|y|\big):=\max_{1\leqslant J\leqslant D}\left(1+\log\frac{1+s+|y|}{1+|c_Js-|y||}\right)
\tag{3.56}
$$

and we use the estimate

$$
\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big|L^\mu Z^\alpha(R+P)\big|\lesssim&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big|L^\mu Z^\alpha\big(\bar\partial^2u\bar\partial^2u\bar\partial^2u\big)\big|\\
\lesssim&\Big(\sum_{|\alpha|\leqslant M_0-2}\big|Z^\alpha\bar\partial^2u\big|\Big)^2\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\big|L^\mu Z^\alpha\bar\partial^2u\big|\\
&+\sum_{|\alpha|\leqslant M_0-2}\big|Z^\alpha\bar\partial^2u\big|\sum_{M_0-1\leqslant|\alpha|\leqslant M-1}\big|Z^\alpha\bar\partial^2u\big|\sum_{\substack{\mu+|\alpha|\leqslant M-M_0+1\\ 1\leqslant\mu\leqslant\mu_0}}\big|L^\mu Z^\alpha\bar\partial^2u\big|\\
&+\sum_{\substack{\mu+|\alpha|\leqslant M-2\\ \mu\leqslant\mu_0-2}}\big|L^\mu Z^\alpha\bar\partial^2u\big|\Big(\sum_{\substack{\mu+|\alpha|\leqslant M-1\\ \mu\leqslant\mu_0-1}}\big|L^\mu Z^\alpha\bar\partial^2u\big|\Big)^2\\
\lesssim&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}}\frac{\varepsilon^2W(s,y)^2}{(1+s+|y|)^2}\big|L^\mu Z^\alpha u\big|\\
&+\sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0}}\frac{(\varepsilon W(s,y))^2}{(1+s+|y|)^2(1+|y|)}\big\|L^\mu Z^\alpha u'\big\|_2\\
&+\frac{\varepsilon^3W(s,y)^3}{(1+s+|y|)(1+|y|)^2}(1+s)^{C(\varepsilon+\sigma)},
\end{aligned}
\tag{3.57}
$$

where we have used (3.2), $(1+|x|)$ version of the pointwise estimate in (3.4), and Lemma 2.17 to derive the last inequality, and the last term can be removed when $\mu_0 = 0$, and C is dependent on C_{M+13,μ_0-1} and C_{M-M_0+14,μ_0}. So that, we obtain

$$B_2 \lesssim \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sum_{1\leqslant I\leqslant D} \varepsilon^2 \sup_{s<t}\left(1+\log\frac{1+s+|y|}{1+|c_I s-|y||}\right)^{-1}\left(1+|y|\right)\left|L^\mu Z^\alpha u_I(s,y)\right| + \varepsilon^2 \sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0}} \left\|L^\mu Z^\alpha u'\right\|_2 + \varepsilon^3(1+t)^{C(\varepsilon+\sigma)}, \tag{3.58}$$

where the last term can be removed when $\mu_0 = 0$, and C is dependent on C_{M+13,μ_0-1} and C_{M-M_0+14,μ_0}, and we have used the bound

$$\frac{z_\sigma(s,|y|)}{(1+s+|y|)^{1-\sigma}} W^3 \lesssim 1. \tag{3.59}$$

The term B_3 is bounded similarly such that

$$B_3 \lesssim \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \sum_{1\leqslant I\leqslant D} \varepsilon^2 \sup_{s<t}\left(1+\log\frac{1+s+|y|}{1+|c_I s-|y||}\right)^{-1}\left(1+|y|\right)\left|L^\mu \partial^\alpha u_I(s,y)\right| + \varepsilon^2 \sum_{\substack{\mu+|\alpha|\leqslant M+7\\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha u'\right\|_2 + \varepsilon^3(1+t)^{C(\varepsilon+\sigma)}, \tag{3.60}$$

where the last term can be removed when $\mu_0 = 0$, and C is dependent on C_{M+17,μ_0-1} and C_{M-M_0+18,μ_0}.

Finally, we obtain for sufficiently small $\varepsilon > 0$

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0}} \left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\left(1+|x|\right)\left|L^\mu Z^\alpha u_I(t,x)\right| \\
&\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M+6\\ \mu\leqslant\mu_0}} \sum_{j\leqslant 1} \left\|\left(\langle y\rangle\partial\right)^j L^\mu Z^\alpha u(0,y)\right\|_{L^2_y} + \sum_{\substack{\mu+|\alpha|\leqslant M+6\\ \mu\leqslant\mu_0}} \left\|\langle y\rangle^{-1/2} L^\mu Z^\alpha u'\right\|^2_{L^2_{t,x}} \\
&\quad + \varepsilon^2 \sum_{\substack{\mu+|\alpha|\leqslant M+3\\ \mu\leqslant\mu_0}} \left\|L^\mu Z^\alpha u'\right\|_2 + \varepsilon^2 \sum_{\substack{\mu+|\alpha|\leqslant M+7\\ \mu\leqslant\mu_0}} \left\|L^\mu \partial^\alpha u'\right\|_2 + \varepsilon^3(1+t)^{C(\varepsilon+\sigma)},
\end{aligned} \tag{3.61}$$

where the last term can be removed when $\mu_0 = 0$, and C is dependent on C_{M+17,μ_0-1} and C_{M-M_0+18,μ_0}.

3.3.7. The estimate for $(1+\log\frac{1+t+|x|}{1+|c_I t-|x||})^{-1}(1+t+|x|)|L^\mu Z^\alpha u_I(t,x)|$

The estimate for

$$\left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\left(1+t+|x|\right)\left|L^\mu Z^\alpha u_I(t,x)\right|$$

follows from the similar and simpler arguments for

$$\left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\bigl(1+|x|\bigr)\bigl|L^\mu Z^\alpha u_I\bigr|.$$

By Proposition 2.6, we have

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}}\left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\bigl(1+t+|x|\bigr)\bigl|L^\mu Z^\alpha u_I(t,x)\bigr|\\
&\quad\lesssim\sum_{\substack{j+\mu+|\alpha|\leqslant M+8\\ \mu\leqslant\mu_0+1}}\sum_{j\leqslant 1}\bigl\|\bigl(\langle y\rangle\partial\bigr)^j L^\mu Z^\alpha u(0,y)\bigr\|_{L^2_y}+\sum_{\substack{\mu+|\alpha|\leqslant M+7\\ \mu\leqslant\mu_0}}\int_0^t\int_{\mathbb{R}^3\setminus\mathcal{K}}\bigl|L^\mu Z^\alpha(B+Q)\bigr|\frac{dy}{|y|}\,ds\\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}}\sup_{(s,y)\in D_I(t,|x|)}|y|\bigl(1+s+|y|\bigr)^{1+\sigma}z_\sigma\bigl(s,|y|\bigr)\bigl|L^\mu Z^\alpha(R+P)(s,y)\bigr|\\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M+4\\ \mu\leqslant\mu_0-1}}\sup_{(s,y)\in(0,t)\times\mathbb{R}^3\setminus\mathcal{K}}|y|\bigl(1+s+|y|\bigr)^{1+\sigma}z_\sigma\bigl(s,|y|\bigr)\bigl|L^\mu\partial^\alpha(R+P)(s,y)\bigr|\\
&\quad=:\sum_{\substack{j+\mu+|\alpha|\leqslant M+8\\ \mu\leqslant\mu_0}}\sum_{j\leqslant 1}\bigl\|\bigl(\langle y\rangle\partial\bigr)^j L^\mu Z^\alpha u(0,y)\bigr\|_{L^2_y}+B_1+B_2+B_3,
\end{aligned}\tag{3.62}$$

where we denote the last three terms by B_1, B_2, and B_3. The term B_1 is bounded by

$$B_1\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M+8\\ \mu\leqslant\mu_0}}\bigl\|\langle y\rangle^{-1/2}L^\mu Z^\alpha u'\bigr\|_{L^2_{t,x}}.\tag{3.63}$$

To bound B_2+B_3, we use the estimates

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M+4\\ \mu\leqslant\mu_0-1}}\bigl|L^\mu Z^\alpha(R+P)\bigr|&\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M+4\\ \mu\leqslant\mu_0-1}}\bigl|L^\mu Z^\alpha\bigl(\bar\partial^2u\bar\partial^2u\bar\partial^2u\bigr)\bigr|\\
&\lesssim\sum_{|\alpha|\leqslant M_0-2}\bigl|Z^\alpha\bar\partial^2u\bigr|\sum_{|\alpha|\leqslant M+4}\bigl|Z^\alpha\bar\partial^2u\bigr|\sum_{\substack{\mu+|\alpha|\leqslant M+4\\ \mu\leqslant\mu_0-1}}\bigl|L^\mu Z^\alpha\bar\partial^2u\bigr|\\
&\quad+\Biggl(\sum_{\substack{\mu+|\alpha|\leqslant (M+4)/2\\ \mu\leqslant\mu_0-2}}\bigl|L^\mu Z^\alpha\bar\partial^2u\bigr|\Biggr)^2\sum_{\substack{\mu+|\alpha|\leqslant M+4\\ \mu\leqslant\mu_0-2}}\bigl|L^\mu Z^\alpha\bar\partial^2u\bigr|\\
&\lesssim\frac{\varepsilon^3W^3}{(1+s+|y|)(1+|y|)^2}(1+s)^{C(\varepsilon+\sigma)},
\end{aligned}\tag{3.64}$$

where W is given by (3.56), and C is dependent on C_{M+19,μ_0-1}. So that, we obtain

$$B_2+B_3\lesssim\varepsilon^3(1+t)^{C(\varepsilon+\sigma)},\tag{3.65}$$

where C is dependent on C_{M+19,μ_0-1}. Therefore we obtain

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant\mu_0-1}}\left(1+\log\frac{1+t+|x|}{1+|c_It-|x||}\right)^{-1}\big(1+t+|x|\big)\big|L^\mu Z^\alpha u_I(t,x)\big|\\
&\lesssim\sum_{\substack{j+\mu+|\alpha|\leqslant M+8\\ \mu\leqslant\mu_0+1}}\sum_{j\leqslant 1}\big\|\big(\langle y\rangle\partial\big)^j L^\mu Z^\alpha u(0,y)\big\|_{L^2_y}+\sum_{\substack{\mu+|\alpha|\leqslant M+8\\ \mu\leqslant\mu_0}}\big\|\langle y\rangle^{-1/2}L^\mu Z^\alpha u'\big\|^2_{L^2_{t,x}}\\
&\quad+\varepsilon^3(1+t)^{C(\varepsilon+\sigma)},
\end{aligned}\tag{3.66}$$

where C is dependent on C_{M+19,μ_0-1}. With M replaced by $M-13$, we obtain the required estimate in (3.4). And we also have the required estimate in (1) in Proposition 3.1.

3.3.8. The proof of (2)

By the Sobolev type estimate Lemma 2.18, we have

$$\begin{aligned}
&\max_{0\leqslant\theta\leqslant 1/2}\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}}\langle x\rangle^{1/2+\theta}\big\langle c_It-|x|\big\rangle^{1-\theta}\big|L^\mu Z^\alpha u_I'(t,x)\big|\\
&\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 2}}\big\|L^\mu Z^\alpha u'(t,x)\big\|_{L^2_x}+\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}}\big\|\big\langle t+|x|\big\rangle L^\mu Z^\alpha(\Box+a\partial_t)u(t,x)\big\|_{L^2_x}\\
&\quad+\sum_{\mu\leqslant 1}(1+t)\big\|L^\mu u'(t,x)\big\|_{L^2(|x|<2)}\\
&=: B_1+B_2+B_3.
\end{aligned}\tag{3.67}$$

The term B_1 is bounded by (3.4). To bound B_2, we use the estimate

$$\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}}\big\{\big|L^\mu Z^\alpha\big(u'u'\big)\big|+\big|L^\mu Z^\alpha(\bar\partial u\bar\partial u\bar\partial u)\big|\big\}\\
&\lesssim\sum_{\substack{\mu+|\alpha|\leqslant (M+2)/2\\ \mu\leqslant 1}}\big|L^\mu Z^\alpha u'\big|\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}}\big|L^\mu Z^\alpha u'\big|\\
&\quad+\Bigg(\sum_{\substack{\mu+|\alpha|\leqslant (M+2)/2\\ \mu\leqslant 1}}\big|L^\mu Z^\alpha\bar\partial u\big|\Bigg)^2\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}}\big|L^\mu Z^\alpha\bar\partial u\big|\\
&\lesssim\frac{\varepsilon W}{1+t+|x|}(1+t)^{C(\varepsilon+\sigma)}\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}}\big|L^\mu Z^\alpha u'\big|\\
&\quad+\left(\frac{\varepsilon W}{1+t+|x|}(1+t)^{C(\varepsilon+\sigma)}\right)^2\Bigg(|u|+\sum_{\substack{1\leqslant\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}}\big|L^\mu Z^\alpha u\big|+\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}}\big|L^\mu Z^\alpha u'\big|\Bigg),
\end{aligned}\tag{3.68}$$

where C is dependent on $C_{(M+2)/2+14,2}$, and moreover we use

$$
\begin{aligned}
&|u| + \sum_{\substack{1\leqslant\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha u\right| + \sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha u'\right| \\
&\quad\lesssim \frac{\varepsilon W(t,x)}{1+t+|x|} + \bigl(1+t+|x|\bigr) \sum_{|\alpha|\leqslant M+1} \left|Z^\alpha u'\right| + \sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha u'\right|. \qquad (3.69)
\end{aligned}
$$

So that, we obtain

$$
B_2 \lesssim \varepsilon^3 + \varepsilon(1+t)^{C(\varepsilon+\sigma)} \sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}} \left\|L^\mu Z^\alpha u'\right\|_{L^2}. \qquad (3.70)
$$

Since the term B_3 is bounded by

$$
B_3 \lesssim (1+t) \sum_{\mu\leqslant 1} \left\|L^\mu u'\right\|_{L^\infty(|x|<2)} \lesssim \varepsilon(1+t)^{C_{15,2}(\varepsilon+\sigma)} \qquad (3.71)
$$

by (3.4), we obtain the required estimate.

3.3.9. The proof of (3)

By the energy estimate (2.2) for $L^\mu Z^\alpha u$, we have

$$
\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \int e_0\bigl(L^\mu Z^\alpha u\bigr)\,dx &\lesssim \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \left\|L^\mu Z^\alpha u'(0,\cdot)\right\|_2^2 + \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}} \left\|L^\mu \partial^\alpha u'\right\|^2_{L^2_{t,x}(|x|<1)} \\
&\quad + \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha u'\right| \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha (\Box + a\partial_t)u\right|\,dx\,ds \\
&=: B_1 + B_2 + B_3. \qquad (3.72)
\end{aligned}
$$

To bound B_2, we use (3.5) to obtain

$$
B_2 \leqslant C_0 \sum_{\substack{j+\mu+|\alpha|\leqslant M+10\\ \mu\leqslant 3}} \sum_{j\leqslant 1} \left\|\bigl(\langle y\rangle\partial\bigr)^j L^\mu Z^\alpha u(0,y)\right\|^2_{L^2_y} + C\varepsilon^4, \qquad (3.73)
$$

where C_0 is independent of A_0, and C is dependent on $C_{M+15,2}$ and $C_{M+21,1}$.

Since the null condition is preserved under the operators L and Z, the term B_3 is bounded by

$$
B_3 \lesssim B_{31} + B_{32} + B_{33}, \qquad (3.74)
$$

where

$$B_{31} := \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha \partial u_I\right| \cdot \left\{ \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant 1}} \left| \sum_{0\leqslant j,k\leqslant 3} \tilde{B}_I^{IIjk} \partial_j L^\mu Z^\alpha u_I \partial_k L^\nu Z^\beta u_I \right| \right.$$
$$\left. + \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\leqslant M\\ \mu+\nu\leqslant 1}} \left| \sum_{0\leqslant j,k,l\leqslant 3} \tilde{Q}_I^{IIjkl} \partial_j L^\mu Z^\alpha u_I \partial_k \partial_l L^\nu Z^\beta u_I \right| \right\} dy\, ds, \tag{3.75}$$

where $\tilde{B}_I^{IIjk}$ and $\tilde{Q}_I^{IIjkl}$ are some constants which satisfy the null conditions (1.9),

$$B_{32} := \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{(J,K)\neq(I,I)} \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha \partial u_I\right| \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha \partial u_J\right|$$
$$\cdot \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha \partial^2 u_K\right| dy\, ds, \tag{3.76}$$

$$B_{33} := \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha u'\right| \Bigg(\sum_{\substack{\mu+|\alpha|\leqslant (M+2)/2\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha u\right| \Bigg)^2$$
$$\cdot \sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha u\right| dy\, ds. \tag{3.77}$$

Using Lemma 2.4, the term B_{31} is bounded by

$$C \int\int \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha \partial u_I\right| \Bigg\{ \frac{1}{\langle y\rangle} \Bigg(\sum_{\substack{\mu+|\alpha|\leqslant M+2\\ \mu\leqslant 2}} \left|L^\mu Z^\alpha u_I\right| \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha \partial u_I\right| \Bigg)$$
$$+ \frac{\langle c_I s - |y|\rangle}{\langle s+|y|\rangle} \Bigg(\sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}} \left|L^\mu Z^\alpha \partial u_I\right| \Bigg)^2 \Bigg\} dy\, ds, \tag{3.78}$$

which is bounded by

$$C\varepsilon \int_0^t (1+s)^{-1/2+\delta} \sum_{\substack{\mu+|\alpha|\leqslant M+1\\ \mu\leqslant 1}} \left\| \langle x\rangle^{-1/2} L^\mu Z^\alpha u' \right\|_2^2 ds \tag{3.79}$$

by the use of the $(1+t+|x|)$ type pointwise estimate in (3.4) and (3.6) with $\theta = 0$ for the first and second terms in the bracket $\{\cdots\}$, respectively. Using the decomposition of the time interval, the above term is bounded by $C\varepsilon^3$ by the weighted energy estimates in (3.4), where the constant C is dependent on $C_{M+15,3}$.

By the use of $\mathbb{R}^3\setminus\mathcal{K} \subset (\mathbb{R}^3\setminus\mathcal{K} \cap \Lambda_I^c) \cup (\mathbb{R}^3\setminus\mathcal{K} \cap \Lambda_J^c)$ for $I \neq J$, and (3.6), the term B_{32} is bounded by (3.79), where C is dependent on $C_{M+6,2}$ and $C_{(M+1)/2+15,2}$.

By the $(1+t+|x|)$ type pointwise estimate in (3.4), the term B_{33} is bounded by

$$C\varepsilon^3 \int_0^t (1+s)^{-1+2\delta} \sum_{\substack{\mu+|\alpha|\leqslant M\\ \mu\leqslant 1}} \big\| \langle x\rangle^{-1/2} L^\mu Z^\alpha u' \big\|_2\, ds, \tag{3.80}$$

where δ is a sufficiently small number, which is bounded by $C\varepsilon^4$, where C is dependent on $C_{M+15,2}$.

Combining these bounds, we obtain the required estimate.

3.3.10. The proof of (4)

By the Sobolev type estimate Lemma 2.18, we have

$$\begin{aligned}
&\max_{0\leqslant\theta\leqslant 1/2} \sum_{|\alpha|\leqslant M} \langle x\rangle^{1/2+\theta} \big\langle c_I t-|x|\big\rangle^{1-\theta} \big|Z^\alpha u_I'(t,x)\big| \\
&\quad\lesssim \sum_{|\alpha|\leqslant M+2} \big\| Z^\alpha u'(t,x)\big\|_{L_x^2} + \sum_{|\alpha|\leqslant M+1} \big\| \big\langle t+|x|\big\rangle Z^\alpha (\Box + a\partial_t) u(t,x)\big\|_{L_x^2} \\
&\qquad + (1+t)\big\|u'(t,x)\big\|_{L^2(|x|<2)}
\end{aligned} \tag{3.81}$$

and we estimate

$$\begin{aligned}
&\sum_{|\alpha|\leqslant M+1} \big\langle t+|x|\big\rangle \big|Z^\alpha(\Box_c + a\partial_t)u(t,x)\big| \\
&\quad\lesssim \big(1+t+|x|\big) \sum_{|\alpha|\leqslant M+2} \big(\big|Z^\alpha\big(u'u'\big)\big| + \big|Z^\alpha(\bar\partial u\bar\partial u\bar\partial u)\big|\big) \\
&\quad\lesssim \varepsilon \sum_{|\alpha|\leqslant M+2} \big|Z^\alpha u'\big| + \frac{(\varepsilon W(t,x))^3(1+t)^{C(\varepsilon+\sigma)}}{(1+t+|x|)(1+|x|)} + \frac{(\varepsilon W(t,x))^2}{1+t+|x|} \sum_{|\alpha|\leqslant M+2} \big|Z^\alpha u'\big| \\
&\qquad + \frac{(\varepsilon W(t,x))^3}{1+t+|x|},
\end{aligned} \tag{3.82}$$

where we have used (3.2), (3.3) and the $(1+|x|)$ type pointwise estimate in (3.4) to derive the last inequality, and C is dependent on $C_{M+13,0}$. And we obtain the required estimate.

3.3.11. The proof of (5)

We consider two smooth functions $0\leqslant\eta\leqslant 1$ and $0\leqslant\chi\leqslant 1$ on $\mathbb{R}$ which satisfy $\eta(r)=0$ for $r\leqslant \min_{1\leqslant I\leqslant D} c_I/10$ and $r\geqslant 10\max_{1\leqslant I\leqslant D} c_I$, $\eta(r)=1$ for $\min_{1\leqslant I\leqslant D} c_I/5\leqslant r\leqslant 5\max_{1\leqslant I\leqslant D} c_I$, $\chi(t)=0$ for $t\leqslant 1$, $\chi(t)=1$ for $t\geqslant 2$. And we define $\rho(t,x)=\chi(t)\eta(|x|/t)$. Then ρ satisfies $\rho=1$ on $\{(t,x)\colon\ t\geqslant 2,\ \min_{1\leqslant I\leqslant D} c_I t/5\leqslant |x|\leqslant 5\max_{1\leqslant I\leqslant D} c_I t\}$, and $\rho=0$ near the t and x axes. We note that ρ satisfies $|L^\mu Z^\alpha\rho|\lesssim 1$.

We decompose $u_I=u_1+u_2+u_3$, where u_1, u_2, u_3 satisfy the following equations:

$$\begin{aligned}
&(\Box_{c_I}+a\partial_t)u_1 = \rho\cdot(B+Q), && u_1|_{\partial\mathcal{K}}=0,\ u_1(0,\cdot)=0,\ \partial_t u_1(0,\cdot)=0,\\
&(\Box_{c_I}+a\partial_t)u_2 = (1-\rho)\cdot(B+Q), && u_2|_{\partial\mathcal{K}}=0,\ u_2(0,\cdot)=u_I(0,\cdot),\ \partial_t u_2(0,\cdot)=\partial_t u_I(0,\cdot),\\
&(\Box_{c_I}+a\partial_t)u_3 = R+P, && u_3|_{\partial\mathcal{K}}=0,\ u_3(0,\cdot)=0,\ \partial_t u_3(0,\cdot)=0.
\end{aligned} \tag{3.83}$$

By Proposition 2.11, we have

$$
\begin{aligned}
&\bigl(1+t+|x|\bigr)\sum_{|\alpha|\leqslant M_0}\bigl|Z^\alpha u_1(t,x)\bigr| \\
&\quad\leqslant C_0\Biggl\{\sum_{\substack{\mu+|\alpha|\leqslant M_0+3\\ \mu\leqslant 1}}\sup_{0\leqslant s\leqslant t}\int\bigl|L^\mu Z^\alpha(\Box_{c_I}+a\partial_t)u_1\bigr|\,dy\left(1+\left|\log\frac{1+t}{1+|c_I t-|x||}\right|\right) \\
&\qquad+\sum_{\substack{\mu+|\alpha|\leqslant M_0+7\\ \mu\leqslant 1}}\sup_{0\leqslant s\leqslant t}\int\bigl|L^\mu Z^\alpha(\Box_{c_I}+a\partial_t)u_1\bigr|\,dy \\
&\qquad+\sum_{|\alpha|\leqslant M_0+3}\sup_{0\leqslant s\leqslant t}(1+s)\bigl\|\partial^\alpha(\Box_{c_I}+a\partial_t)u_1\bigr\|_{L^2(|y|<4)}\Biggr\} \\
&\quad\lesssim\Biggl(A_0^2\varepsilon^2+C_0\sum_{\substack{\mu+|\alpha|\leqslant M_0+8\\ \mu\leqslant 1}}\bigl\|L^\mu Z^\alpha u'\bigr\|_2^2\Biggr)\left(1+\left|\log\frac{1+t}{1+|c_I t-|x||}\right|\right),
\end{aligned}
\tag{3.84}
$$

where we have used

$$
\sum_{\substack{\mu+|\alpha|\leqslant M_0+7\\ \mu\leqslant 1}}\bigl|L^\mu Z^\alpha(\Box_{c_I}+a\partial_t)u_1\bigr|\lesssim\sum_{\substack{\mu+|\alpha|\leqslant M_0+8\\ \mu\leqslant 1}}\bigl|L^\mu Z^\alpha u'\bigr|^2
\tag{3.85}
$$

and (3.3).

By Proposition 2.6, we have

$$
\begin{aligned}
&\bigl(1+t+|x|\bigr)\left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\sum_{|\alpha|\leqslant M_0}\bigl|Z^\alpha u_2(t,x)\bigr| \\
&\quad\lesssim C_0\Biggl\{\sum_{\substack{j+\mu+|\alpha|\leqslant M_0+6\\ \mu\leqslant 1}}\ \sum_{j\leqslant 1}\bigl\|\bigl(\langle x\rangle\partial\bigr)^j L^\mu Z^\alpha u_2(0,\cdot)\bigr\|_2+\sup_{0\leqslant s\leqslant t}(1+s)\sum_{|\alpha|\leqslant M_0+4}\bigl\|\partial^\alpha\bigl(u'u'\bigr)\bigr\|_{L^2(|y|<4)} \\
&\qquad+\sup_{\substack{s\leqslant 2\\ \text{or } |y|\geqslant 10\max_I c_I s\\ \text{or } |y|\leqslant \min_I c_I s/10}}|y|\bigl(1+s+|y|\bigr)^{1+\sigma}z_\sigma(s,y)\sum_{|\alpha|\leqslant M_0+5}\bigl|Z^\alpha\bigl(u'u'\bigr)\bigr|\Biggr\} \\
&\quad=: C_0\{B_1+B_2+B_3\},
\end{aligned}
\tag{3.86}
$$

where we have put the terms in the bracket $\{\cdots\}$ as B_1, B_2, B_3, respectively.

The term B_2 is estimated by

$$
B_2\leqslant C\varepsilon\sup_{s\leqslant t}\sum_{|\alpha|\leqslant M_0+4}\bigl\|\partial^\alpha u'\bigr\|_{L^2(|y|<4)}
\tag{3.87}
$$

by (3.3), where C is dependent on A_0.

The term B_3 is estimated by

$$B_3 \leqslant \left(C_0 \sum_{\substack{\mu+|\alpha|\leqslant M_0+7 \\ \mu\leqslant 1}} \left\| L^\mu Z^\alpha u' \right\|_2 + C_0 \sum_{|\alpha|\leqslant M_0+7} \left\| Z^\alpha u' \right\|_2^2 + C\varepsilon \right)^2 \tag{3.88}$$

by (3.8) with $\theta = 0$, where we have used $\langle c_I s - |y| \rangle \sim 1 + s + |y|$ in the domain of sup, and C is dependent on $C_{M_0+18,0}$. Therefore we obtain

$$C_0(B_1 + B_2 + B_3) \leqslant C_0\varepsilon + C_0 \sum_{\substack{\mu+|\alpha|\leqslant M_0+7 \\ \mu\leqslant 1}} \left\| L^\mu Z^\alpha u' \right\|_2^2 + C_0 \sum_{|\alpha|\leqslant M_0+7} \left\| Z^\alpha u' \right\|_2^4 + C\varepsilon^2 \tag{3.89}$$

by Schwarz inequality, where C_0 is independent of A_0 and C is dependent on A_0.

To bound u_3, we use

$$\begin{aligned} &\sum_{|\alpha|\leqslant M_0} \left| Z^\alpha (\Box_{c_I} + a\partial_t) u_3 \right| + \sum_{|\alpha|\leqslant M_0+4} \left| \partial^\alpha (\Box_{c_I} + a\partial_t) u_3 \right| \\ &\quad \lesssim \left(\sum_{|\alpha|\leqslant M_0/2+4} \left| Z^\alpha u \right| \right)^2 \left(\sum_{|\alpha|\leqslant M_0} \left| Z^\alpha u \right| + \sum_{|\alpha|\leqslant M_0+5} \left| Z^\alpha u' \right| \right) \\ &\quad \lesssim \left(\frac{\varepsilon W(s,y)}{1+s+|y|} \right)^3 + \left(\frac{\varepsilon W(s,y)}{1+s+|y|} \right)^2 \frac{1}{1+|y|} \sum_{|\alpha|\leqslant M_0+7} \left\| Z^\alpha u' \right\|_2, \end{aligned} \tag{3.90}$$

where we have used (3.2) and Lemma 2.17 for the last inequality. So that, by Proposition 2.6, we obtain

$$\begin{aligned} &\big(1 + t + |x|\big) \left(1 + \log \frac{1+t+|x|}{1+|c_I t - |x||} \right) \sum_{|\alpha|\leqslant M_0} \left| Z^\alpha u_3(t,x) \right| \\ &\quad \leqslant C_0 \sum_{\substack{j+\mu+|\alpha|\leqslant M_0+6 \\ \mu\leqslant 1}} \sum_{j\leqslant 1} \left\| \big(\langle x \rangle \partial \big)^j L^\mu Z^\alpha u(0,\cdot) \right\|_2 + C_0 \sum_{|\alpha|\leqslant M_0+7} \left\| Z^\alpha u' \right\|_2^2 + C\varepsilon^3, \end{aligned} \tag{3.91}$$

where C_0 is independent of A_0.

After all, we obtain

$$\begin{aligned} &\big(1 + t + |x|\big) \left(1 + \log \frac{1+t+|x|}{1+|c_I t - |x||} \right)^{-1} \sum_{|\alpha|\leqslant M_0} \left| Z^\alpha u_I(t,x) \right| \\ &\quad \leqslant C_0\varepsilon + C\varepsilon^2 + C_0 \sum_{\substack{\mu+|\alpha|\leqslant M_0+8 \\ \mu\leqslant 1}} \left\| L^\mu Z^\alpha u' \right\|_2^2 \left(1 + \sum_{|\alpha|\leqslant M_0+7} \left\| Z^\alpha u' \right\|_2^2 \right), \end{aligned} \tag{3.92}$$

where C_0 is independent of A_0 and C is dependent on $C_{M_0+18,0}$. Using (3.7), we obtain the required result.

3.3.12. The proof of (6)

We consider the two cases. If $|x| \geqslant \min_{1\leqslant I\leqslant D} c_I t/10$, then we have

$$\begin{aligned}\big(1+t+|x|\big)\sum_{|\alpha|\leqslant M_0-1}\big|Z^\alpha \partial u_I(t,x)\big| &\leqslant C_0\big(1+|x|\big)\sum_{|\alpha|\leqslant M_0-1}\big|Z^\alpha u_I'(t,x)\big| \\ &\leqslant C_0\sum_{|\alpha|\leqslant M_0+1}\big\|Z^\alpha u_I'\big\|_2, \end{aligned} \tag{3.93}$$

where we have used Lemma 2.17. If $|x| < \min_{1\leqslant I\leqslant D} c_I t/10$, then we have

$$\begin{aligned}&\big(1+t+|x|\big)\sum_{|\alpha|\leqslant M_0-1}\big|Z^\alpha \partial u_I(t,x)\big| \\ &\quad\leqslant C_0\big(1+t+|x|\big)\left(1+\log\frac{1+t+|x|}{1+|c_I t-|x||}\right)^{-1}\sum_{|\alpha|\leqslant M_0}\big|Z^\alpha u_I(t,x)\big|, \end{aligned} \tag{3.94}$$

where C_0 is independent of A_0. So that, we obtain the required inequality by the results of (3) and (5).

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